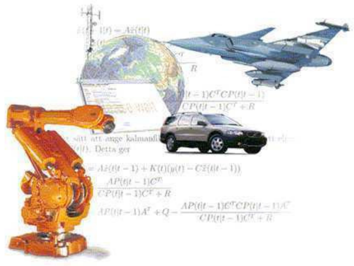
Nonlinear control

Lecture 2



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Nonlinear Control. Lecture 2

- Review of the input-output linearization problem
- Canonical forms and exact state linearization
- Geometry and Lie brackets
- Exact linearization: theory
- Exact linearization: Problems and limitations

Input-output linearization

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

dim y = dim u = m, dim x = n. Assume there exists smallest integer ν_i such that $y_i^{(\nu_i)}$ depends explicitly on u.

- If the decoupling matrix is nonsingular, it is possible to get arbitrary dynamic behavior from r to y.
- Problem: If $\nu_1 + \cdots + \nu_m < n$ there is a subsystem, not visible from y (the zero dynamics).
- If the zero dynamics is unsuitable (e.g. unbounded variables) the input-output linearization does not work.

Taking away the zero dynamics

- Redefine the output so that the relative degree satisfies $\nu_1 + \cdots + \nu_m = n$.
- Then there is no zero dynamics.
- **The scalar case:** Find a new output $\eta = \phi(x)$ so that $\eta, \dot{\eta}, \ldots, \eta^{(n-1)}$ do not directly depend on u.

Controller canonical form

The scalar case, cont'd.
Introduce new variables

$$z_1 = \eta = \phi(x), \quad z_2 = \dot{\eta}, \dots, z_n = \eta^{(n-1)}$$

The state description then becomes

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= f_n(z) + g_n(z)u \end{aligned}$$

controller canonical form

Controller canonical form makes control trivial

Select the state feedback

$$u = \frac{1}{g_n(z)}(-f_n(z) + \tilde{f}_n(z) + \tilde{g}_n(z)r)$$

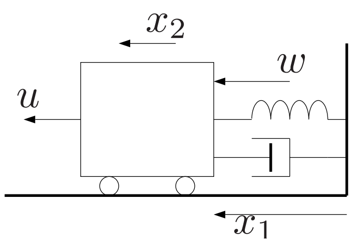
The resulting dynamics is

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= \tilde{f}_n(z)(z) + \tilde{g}_n(z)r \end{aligned}$$

with $\tilde{f}_n$ and $\tilde{g}_n$ arbitrary.

Mechanical systems

Mechanical systems often have the controller canonical form automatically.



$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k(x_1) - b(x_2) + u \end{aligned}$$

where k is a nonlinear spring force and b a nonlinear damping.

Getting relative degree = n

If $z_1 = h(x)$ then

$$\begin{aligned} \dot{z}_1 &= L_f h + uL_g h && \rightarrow L_g h = 0 \\ \ddot{z}_1 &= L_f^2 h + uL_g L_f h && \rightarrow L_g L_f h = 0 \\ &\vdots \\ z_1^{(n-1)} &= L_f^{n-1} h + uL_g L_f^{n-2} h && \rightarrow L_g L_f^{n-2} h = 0 \\ z_1^{(n)} &= L_f^n h + uL_g L_f^{n-1} h && \rightarrow L_g L_f^{n-1} h \neq 0 \end{aligned}$$

h has to satisfy a system of PDEs.

Simplifications

In the expression

$$L_f L_g - L_g L_f$$

the second order derivatives cancel. Defining the **Lie bracket**:

$$[f, g] = g_x f - f_x g$$

one can write

$$L_f L_g - L_g L_f = L_{[f,g]}$$

Notation for iterated Lie brackets:

$$(\text{ad}^0 f, g) = g, \ (\text{ad}^1 f, g) = [f, g], \ (\text{ad}^2 f, g) = [f.[f.g]], \ \text{etc}$$

Where does all the Lie stuff come from?

Photo: Ludwik Szacinski

Sophus Lie (1842-1899)
Norwegian mathematician

Lie derivatives, Lie brackets, Lie groups, Lie algebras,. . .

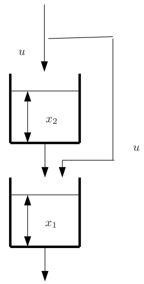
Getting controller canonical form, cont'd

If $z_1 = h(x)$ then h has to satisfy

$$\begin{aligned} L_g h &= 0 \\ L_{[f,g]} h &= 0 \\ &\vdots \\ L_{(ad^{n-2} f, g)} h &= 0 \\ L_{(ad^{n-1} f, g)} h &\neq 0 \end{aligned}$$

This is a system of first order PDEs.

A tank example



$$\begin{aligned} \dot{x}_1 &= -\sqrt{x_1} + \sqrt{x_2} + u \\ \dot{x}_2 &= -\sqrt{x_2} + u \end{aligned}$$

The function h has to satisfy

$$\frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_2} = 0$$

A solution is $h = x_1 - x_2$

An academic example

$$\dot{x}_1 = x_2 + x_3 u$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$

h has to satisfy

$$x_3 \frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_3} = 0$$
$$\frac{\partial h}{\partial x_2} = 0$$

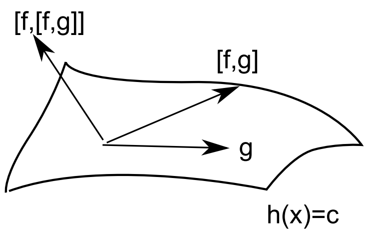
A solution is $h = x_1 - \frac{x_3^2}{2}$

Solvability – the question

When is it possible to solve

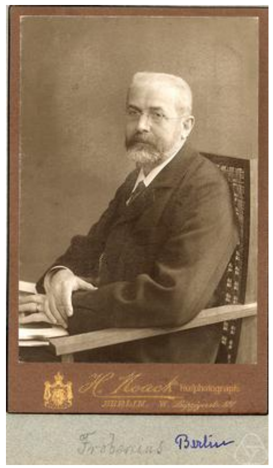
$$L_g h = 0$$
$$L_{[f,g]} h = 0$$
$$\vdots$$
$$L_{(ad^{n-2} f,g)} h = 0$$
$$L_{(ad^{n-1} f,g)} h \neq 0$$

Geometric interpretation, $n = 3$



The surface $h(x) = c$ has to be tangent to g and $[f, g]$ and non-tangent to $[f, [f, g]]$ at all points.

Solvability – the answer



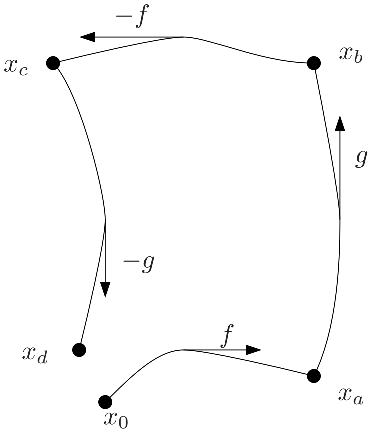
Georg Frobenius (1849-1917):
Solvable in neighborhood of x_0 iff

$g, [f, g], \ldots, (\text{ad}^{n-1} f, g)$ lin. indep. at x_0

$g, [f, g], \ldots, (\text{ad}^{n-2} f, g)$ involutive in nbh x_0

Involutive: Lie brackets of elements give no new elements (just linear combinations of existing ones)

Interpretation of Lie bracket



Following
f for $t \in [0, h]$,
g for $t \in [h, 2h]$,
$-f$ for $t \in [2h, 3h]$,
$-g$ for $t \in [3h, 4h]$,
the total distance traveled is

$$x_d - x_0 = h^2 \; [f, g](x_0) + O(h^3)$$

The multivariable case

For general m one has to consider

$$\begin{aligned} G_0 &= \text{span}(g_1, \ldots, g_m) \\ G_1 &= \text{span}(g_1, \ldots, g_m, [f, g_1], \ldots, [f, g_m]) \\ &\vdots \\ G_{n-1} &= \text{span}((\text{ad}^k f, g_j), \; k = 0, \ldots, n-1, \; j = 1, \ldots, m) \end{aligned}$$

If $g(x_0)$ has rank m, then it is possible to satisfy $\nu_1 + \cdots + \nu_m = n$ in nbh of x_0 iff
- ☐ all G_i have constant rank near x_0
- ☐ G_{n-1} has rank n.
- ☐ $G_0, \ldots, G_{n-2}$ are involutive.

The decoupling matrix revisited

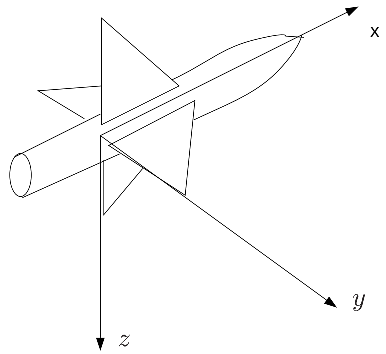
$$R(x)_{ij} = L_{g_j} L_f^{\nu_i - 1} h_i$$

Nonsingularity of R implies full rank of the matrices in the product

$$\underbrace{\begin{bmatrix} (h_1)_x \\ \vdots \\ (L_f^{\nu_1 - 1} h_1)_x \\ \vdots \\ (h_m)_x \\ \vdots \\ (L_f^{\nu_m - 1} h_m)_x \end{bmatrix}}_{S(x)} \underbrace{[g_1, \ldots, g_m, \ldots, (ad^{r-1} f, g_1), \ldots, (ad^{r-1} f, g_m)]}_{T(x)}$$

$r = \max \nu_i$

Missile control



x_1: (velocity, z)/(velocity, x)
x_2: (velocity, y)/(velocity, x)
x_3, x_4, x_5 angular velocities along x, y, z
u_1, u_2, u_3 elevator, rudder, aileron

Model of missile

$$\dot{x}_1 = x_4 - x_3x_2 + F_1(x) + g_{11}u_1$$
$$\dot{x}_2 = -x_5 + x_3x_1 + F_2(x) + g_{22}u_2$$
$$\dot{x}_3 = F_3(x) + g_{33}u_3$$
$$\dot{x}_4 = I_q x_3x_5 + F_4(x) + g_{41}u_1$$
$$\dot{x}_5 = -I_q x_3x_4 + F_5(x) + g_{52}u_2$$

F_i: aerodynamic forces and torques.
Always exactly linearizable if all g_{ij} constant.
Coordinate change: $z_1 = x_1 - \gamma_1 x_4$, $z_2 = x_2 - \gamma_2 x_5$, $\gamma_1 = g_{11}/g_{41}$, $\gamma_2 = g_{22}/g_{52}$, $z_3 = x_3$

Controller canonical form

Advantage: The dynamics can be changed “into anything”.

Limitations:

- Can not always be obtained
- Transformation might be impossible to calculate explicitly
- Transformation might not be global
- Transformation might have singularities

Exact state linearization

Using the controller canonical form and feedback to get a linear system.
Advantage: Linear systems are well known

Disadvantages:

- Linear dynamics might not be the best thing.
- Might remove “good” nonlinearities.
- Control saturation might cause problems.
- Robustness is not well understood.