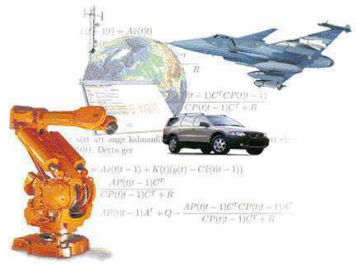
Nonlinear control

Lecture 4: Lyapunov theory



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The course so far

Various transformations to make design easier. Lie derivatives and Lie brackets. "Geometric control theory".

- Input-output linearization.
  - Decoupling and input-output dynamics easy.
  - Requires "good" zero dynamics.
- Controller form.
  - State feedback to achieve arbitrary dynamics easy.
  - Fairly restrictive conditions
  - Often computationally difficult
- Observers
  - High gain observers
  - Transformations to get linear error dynamics
  - Differentiation and equation solving

Observers with linear error dynamics – extensions

The transformations can only be carried out under very restrictive conditions. There are however many extensions in the litterature:

- Allow nonlinear transformations of the output. (Covered in the lecture notes).
- Allow transformation of the time variable.
- Embed the dynamics an a larger state space.
- ....

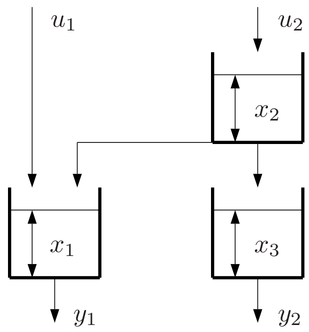
Differentiation and equation solving

Idea: Compute x from the nonlinear system of equations

$$y_1 = h_1, \quad \dot{y}_1 = L_f h_1, \quad \dots \quad y^{(\sigma_1-1)} = L_f^{\sigma_1-1} h_1$$
$$\vdots$$
$$y_p = h_p, \quad \dot{y}_p = L_f h_p, \quad \dots \quad y^{(\sigma_p-1)} = L_f^{\sigma_p-1} h_p$$

- How do you get $\dot{y}_i$, $\ddot{y}_i$ etc.? – Use approximate differentiation e.g. $\frac{1}{1+sT}$
- How do you solve the nonlinear equations? Problem dependent.

A simple example



A model is

$$\dot{x}_1 = -\sqrt{x_1} + \sqrt{x_2} + u_1$$
$$\dot{x}_2 = -2\sqrt{x_2} + u_2$$
$$\dot{x}_3 = -\sqrt{x_3} + \sqrt{x_2}$$
$$y_1 = x_1$$
$$y_2 = x_3$$

$\sigma_1 = 2, \sigma_2 = 1$ ($\sigma_1 = 1, \sigma_2 = 2$ also works)

$$x_1 = y_1, \quad x_2 = (\dot{y}_1 + \sqrt{y_1} - u)^2, \quad x_3 = y_2$$

Separation theorems

Separation theorem: Good state feedback + good observer gives good overall performance.

- Linear systems:
  - Pole placement of state feedback + pole placement of observer gives overall pole placement
  - LQG: optimal feedback + Kalman filter gives overall optimality
- Nonlinear systems: few and very restrictive results.

Origin of stability theory

source: Wikimedia Commons

Alexander Lyapunov
(1857 – 1918)
Russian mathematician and physicist.
Almost all known stability theory for nonlinear systems goes back to him.

Lyapunov theory

Given $\dot{x} = f(x)$. Find a scalar function V, **Lyapunov function**, which is decreasing along solutions:

$$\dot{V} = V_x f \leq 0$$

The solution remains in a set

$$B_d = \{x : V(x) \leq d\}$$

once it enters it.
If V is **radially unbounded**, i.e.

$$V(x) \to \infty \text{ as } |x| \to \infty$$

then the B_d-sets are bounded, i .e. solutions are always bounded.

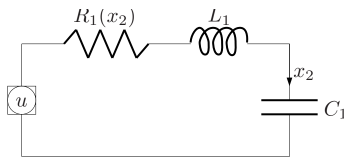
Convergence

Once we know that a solution stays within a B_d-set, where can it converge to?

$$E = \{x \in B_d : V_x(x)f(x) = 0\}$$
$$M = \text{largest invariant set in } E$$

All solutions converge to points in M.

An electrical circuit with nonlinear resistance



x_1: voltage of capacitor. $R(0) = 0$, $x_2 R(x_2) > 0$ if $x_2 \neq 0$.

$$C\dot{x}_1 = x_2$$
$$L\dot{x}_2 = -x_1 - R(x_2) + u$$

Autonomous system ($u = 0$):

$$V = \frac{Cx_1^2}{2} + \frac{Lx_2^2}{2} \quad \rightarrow \quad \dot{V} = -x_2 R(x_2) \leq 0$$

$$E = \{x : x_2 = 0\}, \quad M = \{0\}$$

All solutions converge to the origin.

Computations

How do you compute Lyapunov functions?

1. $\dot{x} = Ax$,
take $V = x^T P x$, solve $A^T P + PA = -Q$

2. $\dot{x} = f(x)$ with asymptotically stable linearization.
Compute V using a series expansion, by identifying coefficients.

3. Solve the first order PDE $V_x f = -g$, where g is positive definite.

4. Use physics (energy balance).

Electrical example, continued

With nonzero u and output $y = x_2$:

$$\dot{V} = -x_2 R(x_2) + x_2 u = -x_2 R(x_2) + yu$$

Integrating both sides gives

$$\underbrace{\int_0^T uy\, dt}_{\text{energy put into system}} = \underbrace{\int_0^T x_2 R(x_2) dt}_{\text{energy dissipated}} + \underbrace{V(x(T)) - V(x(0))}_{\text{change in stored energy}}$$

From this energy balance follws that

$$\int_0^T uy\, dt + V(x(0)) \geq 0$$

for any time interval T and any input u.
A system having this property is called *passive*.

Passivity and dissipativity

Dissipative system:

$$\int_0^T w(u(t), y(t))\, dt + \gamma(x(0)) \geq 0$$

for some w ($w(0,0) = 0$) and some nonnegative γ.

$$\dot{x} = f(x, u), \quad y = h(x, u), \quad f(0,0) = 0, \quad h(0,0) = 0$$

Special cases:

- $w(u, y) = -y^T y + k^2 u^T u$: **finite gain**
- $w(u, y) = u^T y$: **passive**. If u, y is an effort/flow pair the system absorbs energy.
- $w(u, y) = u^T y - \epsilon u^T u$: **input strictly passive** ($\epsilon > 0$)
- $w(u, y) = u^T y - \epsilon y^T y$: **output strictly passive** ($\epsilon > 0$)

Finite gain from Lyapunov function

Consider a nonlinear system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

satisfying $f(0) = 0, \quad h(0) = 0, \quad |g(x)| \leq c_1, \quad |h(x)| \leq c_2|x|$

Let the open loop system $\dot{x} = f$ have a Lyapunov function V:
$V(0) = 0, \quad V_x(x)f(x) \leq -c_3|x|^2, \quad |V_x(x)| \leq c_4|x|$

Then there is finite gain from input to output

$$\int_0^T y^T(t)y(t)dt \leq k^2 \int_0^T u^T(t)u(t)dt + \gamma(x(0))$$

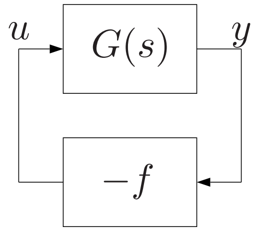
From dissipativity to Lyapunov functions

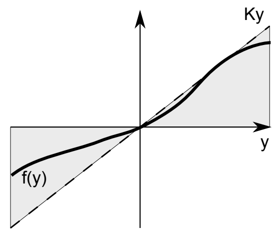
storage function V :

$$V(x(0)) = -\inf_{u,T} \int_0^T w(u(t), y(t))\, dt$$

Connected to optimal control.

Linear block, nonlinear feedback





$u = -f(t, y)$ $\qquad$ $f(t, y)(f(t, y) - Ky) \leq 0$

Multivariable generalization (K is a matrix):

$$f(t, y)^T (f(t, y) - Ky) \leq 0, \quad K > 0$$

Lyapunov conditions

$$V = x^T P x$$

is a Lyapunov function if the following matrix equations can be solved

$$A^T P + PA + L^T L = -\epsilon P$$
$$PB = C^T K - \sqrt{2} L^T$$

Frequency domain interpretation: define

$$G_K(s) = K \underbrace{C(sI - A)^{-1} B}_{G(s)} + I$$

If the following factorization is possible

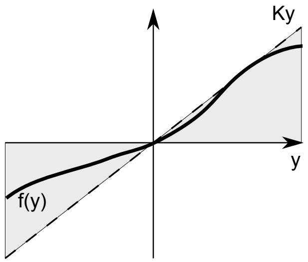
$$G_K(s - \epsilon/2) + G_K^T(-s - \epsilon/2) = V^T(-s) V(s)$$

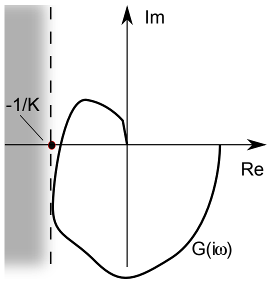
then the matrix equations are solvable (one form of the Kalman-Yakubovich-Popov lemma).

Kalman-Yakubovich-Popov

- Rudolf (Rudy) Emil Kalman (1930–). Born Hungary. University of Florida, Gainsville. ETH, Zürich. Kalman filter. Kalman decomposition. Observability. Controllability. Linear quadratic control.
- Vladimir Andreevich Yakubovich (1926–). Born Novosibirsk. Saint Petersburg State University. Nonlinear stability theory. Optimal control. Adaptive systems.
- Vasile Mihai Popov (1928–). Born Romania. Bucharest Polytechnic Institute. University of Florida, Gainesville. Stability theory for nonlinear systems. Passivity. Canonical forms.

Stability condition, scalar case





$$f(t,y)(f(t,y) - Ky) \le 0$$

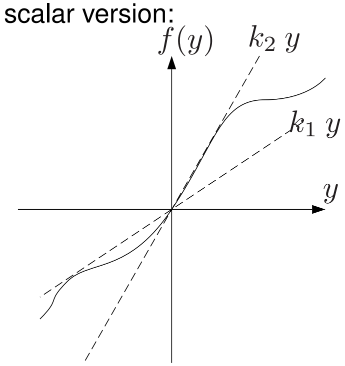
Stability is guaranteed by the condition

$$\text{Re}\, G_K(i\omega) > 0 \quad \Leftrightarrow \quad \text{Re}\, G(i\omega) > -1/K$$

G_K is said to be *positive real*.

Extension 1: lower bounds

scalar version:

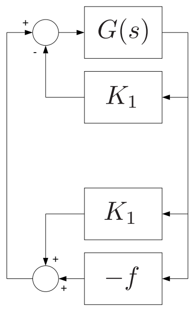


multivariable version:

$$(f(t,y) - K_1 y)^T (f(t,y) - K_2 y) \le 0$$

A standard trick: Pole shifting

Add and subtract a linear signal path:

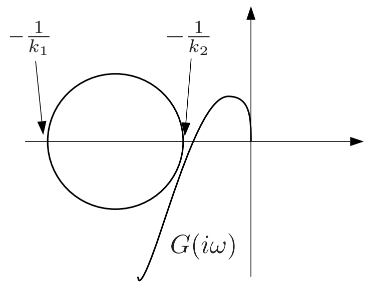


The system G with feedback f is equivalent to $G(I + K_1 G)$ having the feedback $f - K_1 y$.

Result from pole shifting

If $(I + K_2 G(s)(I + K_1 G(s))^{-1}$ is strictly positive real then the Lyapunov function can be constructed

Scalar case:



The circle criterion: G does not enter or encircle the circle.

Extension 2: Popov criterion

Time invariant f, $f^T(y)(f(y) - Ky) \leq 0$

Try Lyapunov function

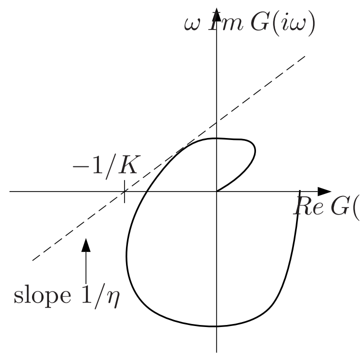
$$V = x^T P x + 2\eta \int_0^y f^T(\sigma) K \, d\sigma$$

(requires $f^T K$ to be a gradient)

multi-variable Popov criterion:
$I + (1 + \eta s) K G(s)$ is strictly positive real
for some $\eta \geq 0$, $-1/\eta$ not eigenvalue of A.
$\Rightarrow V$ guarantees stability.

Popov criterion, SISO

SISO case, the **classical Popov criterion**:

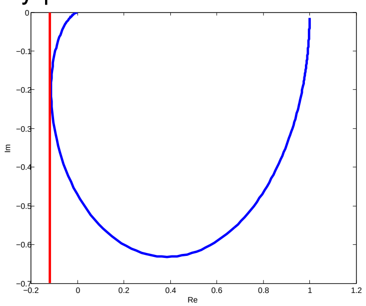


V is a Lyapunov function if the Popov curve lies below the straight line.

Example

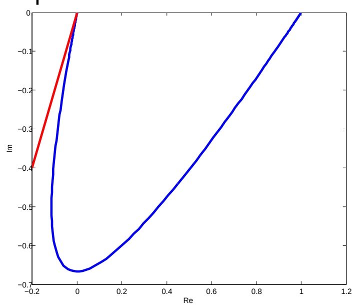
$G(s) = \frac{2}{(s+1)(s+2)}$ with saturated P-controller:

Nyquist curve:



$0 < K < 8.7$ guarantees stability

Popov curve:



any $K > 0$ guarantees stability