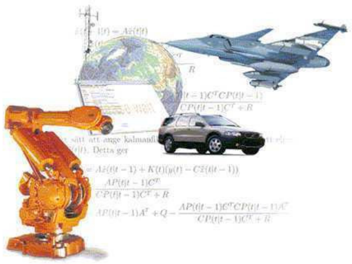
# Nonlinear control

## Lecture 6. Optimal control



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## Review of course

- Geometric control theory
  - input-output linearization
  - controller canonical form
  - observer canonical form
- Lyapunov theory
  - Stability results
  - Lyapunov design: back-stepping etc.



## Optimal control

The optimal control problem:

$$\dot{x} = f(t,x,u), \quad u \in U, \quad \text{physical system}$$
$$x(t_0) = x_0, \quad (t_1, x(t_1))) \in M \quad \text{End point conditions}$$
$$\text{minimize } J = \int_{t_0}^{t_1} L(t,x(t),u(t))dt + \phi(t_1, x(t_1)), \quad \text{Criterion}$$

Note:

- $u$ constrained.
- Initial point and time fixed.
- Final point and time constrained.



## The time invariant infinite horizon problem

Often the optimal control problem is specified for a time-invariant system over an infinite time interval.

$$\dot{x} = f(x,u), \quad u \in U, \quad x(0) = x_0$$
$$\text{minimize } J(x_0, u(.)) = \int_0^\infty L(x(t),u(t))dt$$

Assume that

- The origin is an equilibrium:

$$f(0,0) = 0$$

- Zero is an allowed control signal: $0 \in U$
- Deviation from the equilibrium is penalized in the criterion:

$$L(0,0) = 0, \quad L(x,u) \geq 0$$

## Optimal return

The optimal return function

$$V(x) = \min_{u(.)} J(x, u(.))$$

satisfies the principle of optimality

$$V(x) \leq \int_0^h L(x(t), u(t))dt + V(x(h))$$

with equality when $u$ is optimal.



## The Hamilton-Jacobi-Bellman equation

Suppose the optimal return function $V$ is continuously differentiable.

Then $V$ satisfies the *Hamilton-Jacobi-Bellman equation* (HJB).

$$0 = \min_{u \in U}(V_x(x)f(x, u) + L(x, u))$$



## Sufficient conditions for optimality

Suppose $W$ satisfies HJB:

$$0 = \min_{u \in U} (L(x, u) + W_x(x)f(x, u))$$

and that the minimizing control is $u = k(x)$.

Also assume that $u = k(x)$ drives the state to zero asymptotically.

Then $u = k(x)$ is the optimal control law (compared to all other controls that also drive the state to zero asymptotically) and $W$ is the corresponding optimal return function.



## Lyapunov theory and optimal control

Since $L$ is positive $V$ is a Lyapunov function for the closed loop system:

$$V_x(x)f(x, k(x)) = -L(x, k(x))$$

Follows from HJB.

## A simple example

System: $\dot{x} = u, \quad |u| \le 1$
Criterion: $\int_0^\infty (x^2 + u^2) dt$

The HJB equation: $0 = \min_{|u| \le 1} (x^2 + u^2 + V_x u)$

Solution:

$$V(x) = \begin{cases} x^2, & |x| \le 1 \\ |x|^3/3 + |x| - 1/3, & |x| > 1 \end{cases}$$

Control law:

$$u = \begin{cases} -x, & |x| \le 1 \\ -sgn(x), & |x| > 1 \end{cases}$$



## A DC motor with bounded input

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u, \quad |u| \le 1$$

Criterion:

$$J = \int_0^\infty \left( x_1^2 + ru^2 \right) dt$$

The HJB equation is

$$0 = \min_{|u| \le 1} \left( x_1^2 + ru^2 + x_2 V_{x_1} + (-x_2 + u) V_{x_2} \right)$$



## DC motor. Solution

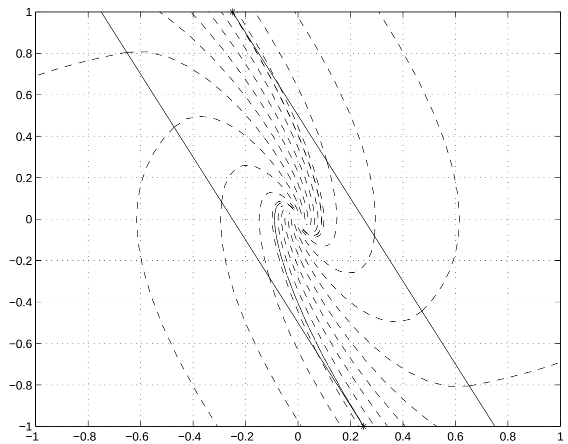
Close to the origin we have a standard linear quadratic control problem. With $r = 1/16$ the solution is then

$$V = \frac{3}{4}x_1^2 + \frac{1}{2}x_1 x_2 + \frac{1}{8}x_2^2, \quad u = -4x_1 - 2x_2$$

To get a solution for all values of $x$ numerical computations are necessary.

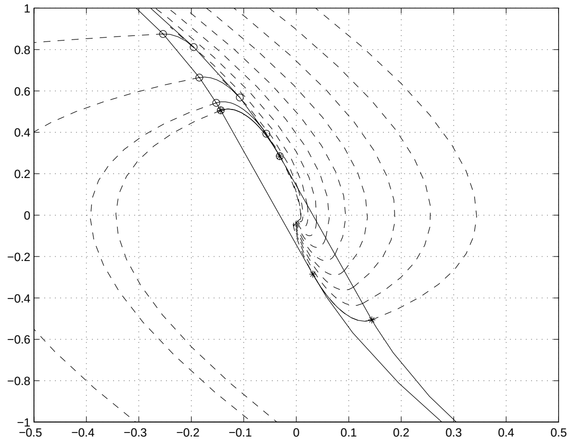


## DC motor. $r = 0.0625$




Solid lines: $|u| = 1$, dashed lines: optimal trajectories

# DC motor. $r = 0.001$




Solid lines: $|u| = 1$, dashed lines: optimal trajectories



# Control affine with quadratic cost

$$\dot{x} = a(x) + b(x)u, \quad J = \int_0^\infty \left( \ell(x) + \frac{1}{2} u^T R u \right) dt$$

The optimal control is

$$u = k(x) = -R^{-1} b(x)^T V_x(x)^T$$

where $V$ is given by

$$0 = \ell(x) + V_x(x) a(x) - \frac{1}{2} V_x(x) b(x) R^{-1} b(x)^T V_x(x)^T$$



# Inherent robustness

If $L$ has the form

$$L(x,u) = \ell(x) + r(u), \quad \ell(0) = 0,\ r(0) = 0, \quad \ell(x) \geq 0,\ r(x) > 0, x \neq 0$$

and if $z^T f(x,u)$ is monotone in $u$ for all $z$, then the optimal control has an *infinite gain margin*:

When replacing $k(x)$ with $\phi(k(x))$, where
$z\phi(z) \geq z^2$
$V$ remains a Lyapunov function.
There are many variations on this result, see e.g.
S.T.Glad: On the Gain Margin of Nonlinear and Optimal Regulators. IEEE Transactions on Automatic Control. AC-29(7):615-620. 1984.



# Solving

How do you solve optimal control problems?

- Explicit analytical solution. (very few problems)
- Series expansion. Only local solutions.
- PDE solvers for Hamilton-Jacobi equation
- Adjusting parameters in given structure.
- Model predictive control (MPC).

## A series solution of the HJB equation

Assume series expansions

$$a(x) = Ax + a^{(2)}(x) + \cdots, \quad b(x) = B + b^{(1)}(x) + \cdots$$

$$\ell(x) = \frac{1}{2}x^T Q x + \ell^{(3)}(x) + \cdots, \quad V(x) = \frac{1}{2}x^T S x + V^{(3)}(x) + \cdots$$

The HJB equation

$$0 = l(x) + V_x(x)a(x) - \frac{1}{2}V_x(x)b(x)R^{-1}b(x)^T V_x(x)^T$$

then takes the form

$$0 = Q + A^T S + SA - SBR^{-1}B^T S$$

$$-(V^{(m)})_x A_c x = l^{(m)}(x) + f(V^{(2)}, \ldots, V^{(m-1)}), \quad m \geq 3$$

where $A_c = A - BR^{-1}B^T S$.
The higher order coefficients of $V$ can be computed from *linear* equations.



## Extensions to optimal control

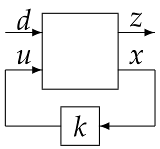
- Controllability and observability Gramians for nonlinear systems.
- Model reduction
- Optimal control of differential algebraic equations
- Criteria with discounted cost.

Johan Sjöberg: Optimal control and model reduction of nonlinear DAE models. Linköping studies in science and technology; Dissertations 1166.



## Nonlinear robustness

**Nonlinear robustness techniques** “Nonlinear $H_\infty$”




$$\dot{x} = f(x) + g(x)u + b(x)d, \quad z = h(x)$$

$$y = \begin{bmatrix} z \\ u \end{bmatrix}$$

Find $u = k(x)$ to make the gain from $d$ to $y$ smaller than $\gamma$.



## The Hamilton-Jacobi inequality

Let $\gamma$ be given. If the *Hamilton-Jacobi inequality*

$$V_x f + \frac{1}{4}V_x\left(\frac{1}{\gamma^2}bb^T - gg^T\right)V_x^T + h^T h \leq 0, \quad V(x_0) = 0$$

has a nonnegative solution $V$, then the control law

$$u = k(x), \quad k(x) = -\frac{1}{2}g(x)^T V_x(x)^T$$

gives a gain from $d$ to $y$ which is less than or equal to $\gamma$,