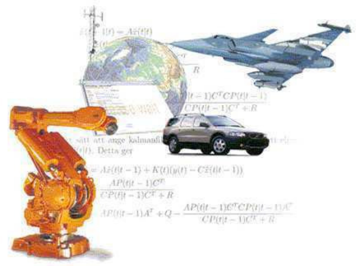
Nonlinear control

Lecture 7



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Review of course

- Geometric control theory
  - input-output linearization
  - controller canonical form
  - observer canonical form
- Lyapunov theory
  - Stability results
  - Lyapunov design: back-stepping etc.
- Design methods
  - Optimal control
  - Extensions of H_∞

Mechanical systems

Mechanical system described by coordinates q (typically positions and angles)

Often the energy is given by an expression like

$$H(q, \dot{q}) = \underbrace{\frac{1}{2}\dot{q}^T D(q)\dot{q}}_{T=\text{kinetic energy}} + \underbrace{V(q)}_{\text{potential energy}}$$

Let u_k be force or torque along coordinate q_k, so that $u_k\dot{q}_k$ is the power absorbed into the system.

Lagrange's equation

For a wide range of physical systems the dynamics is given by

$$\frac{d}{dt}L_{\dot{q}}^T - L_q^T = -F(\dot{q}) + Bu$$

where

- $L(q, \dot{q}) = T(q, \dot{q}) - V(q) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q)$
- F is a generalized force satisfying $\dot{q}^T F(\dot{q}) \geq 0$
- u is the control signal

Passivity of a Lagrangian system

With

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad y = \dot{q}$$

the system is passive with energy balance

$$\int_0^T u^T y \, dt + H(x(0)) - H(x(T)) = \int_0^T \dot{q}^T F dt \geq 0$$

where the right hand side is the energy dissipation.
With sufficient dissipation the system will come to rest with $\dot{q} = 0$ at a minimum of $V(q)$.
H is a natural Lyapunov function.

Lyapunov function for Lagrangian system

Lagrangian system with $u = 0$. Assume:

- The potential V is radially unbounded.
- The kinteic energy satisfies $T \geq \epsilon_1 \dot{q}^T \dot{q}, \quad \epsilon_1 > 0$.
- There is precisely one point q_o such that $V_q(q_o) = 0$
- The dissipative force satisfies $\dot{q}^T F(\dot{q}) \geq \epsilon_2 \dot{q}^T \dot{q}, \quad \epsilon_2 > 0$.

Then the equilibrium $q = q_o, \dot{q} = 0$ is globally asymptotically stable with Lyapunov function $H = T + V$.

Design objectives

There are typically two things one wants to change in the physical system:

- The equilibrium. The natural one is usually not the desired one (given e.g. by a reference signal)
- The damping. Usually it is too low. Also the distribution might be important.

Energy shaping

Suppose the desired position is q_r. Then the control law

$$u = V_q(q)^T + K_p(q_r - q) - K_d \dot{q} + v, \quad v \text{ external signal}$$

with K_p, K_d positive definite, gives the energy balance

$$\int_0^T v^T y \, dt + \tilde{H}(x(0)) - \tilde{H}(x(T)) = \int_0^T (\dot{q}^T F + \dot{q}^T K_d \dot{q}) \, dt$$

where the new energy function is

$$\tilde{H} = \frac{1}{2} \dot{q}^T D(q) \dot{q} + \frac{1}{2}(q - q_r)^T K_p (q - q_r)$$

with a minimum at $q = q_r$, $\dot{q} = 0$ and increased damping.

Fully actuated systems

We have seen that mechanical systems can be controlled using PD-control plus compensation for potential energy (“gravity compensation”).
Typically u has to have the same dimension as q, “fully actuated system”.

Storage elements

Many physical systems are modeled using the following concepts:

- There is a stored quantity x (e.g.electric charge)
- There is a flow $f = \dot{x}$ (e.g. electric current)
- The stored energy is $H(x)$ (e.g. $\frac{1}{2C}x^2$ for a capacitor)
- There is an effort $e = \frac{dH}{dx}$ (e.g. voltage)
- The power absorbed into the system is thus $\frac{d}{dt}H(x) = \frac{dH}{dx}\dot{x} = ef$

Connected storage elements

Many engineering systems are built from components so that the following is true (e.g. simple electric circuits, bond graphs)

- There are n storage elements. The i:th element has storage variable x_i, flow f_i, effort e_i
- The stored variables, flows and efforts are collected into vectors x, f and e.
- The stored energy in component i is $H_i(x_i)$.
- The total energy is $H(x) = H_1(x_1) + \cdots + H_n(x_n)$
- The rules for connecting components (Kirchhoff’s laws, s- and p-junctions) give a relation $f = Me$.
- The matrix M is skew-symmetric: $M = -M^T$ (power is preserved outside the components)

Interconnected systems are Hamiltonian

The interconnected systems we have described are described by

$$\dot{x} = f = Me = MH_x(x)^T, \quad H_x = (\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n})$$

with M skew-symmetric. Systems of this form are called *Hamiltonian* with *Hamilton function* H.
Since $\dot{H} = H_x\dot{x} = H_xMH_x^T = 0$ the total energy is constant.

Inputs and outputs

Now assume that some efforts and flows are not connected to storage elements but are inputs and outputs. Partition the vectors and M as

$$e = \begin{bmatrix} e_x \\ e_u \end{bmatrix}, \quad f = \begin{bmatrix} f_x \\ f_y \end{bmatrix}, \quad M = \begin{bmatrix} M_{xx} & M_{xu} \\ -M_{ux}^T & 0 \end{bmatrix}$$

where e_x, f_x are connected to storage elements, $e_u = u$ is the input and $f_y = -y$ is the output. The system description is then

$$\dot{x} = M_{xx}e_x + M_{xu}e_u = M_{xx}H_x^T(x) + M_{xu}u$$
$$y = -f_y = M_{ux}^T H_x^T(x)$$

Port controlled Hamiltonian systems

The system on the previous slide is an example of a *port controlled Hamiltonian system*. In general the description is

$$\dot{x} = J(x)H_x^T(x) + g(x)u$$
$$y = g^T(x)H_x^T(x)$$

where $J(x)$ is skew-symmetric.

Dissipation

Dissipation can be introduced by elements having relations like $e_i = Rf_i$ (e.g. an electrical resistor)
This can be modeled by first taking a port-controlled Hamiltonian system of the form

$$\dot{x} = J(x)H_x^T(x) + g(x)u + g_R(x)u_R$$
$$y = g^T(x)H_x^T(x)$$
$$y_R = g_R^T(x)H_x^T(x)$$

and then setting $u_R = -\bar{R}y_R$. The model is then

$$\dot{x} = (J(x) - R(x))H_x^T(x) + g(x)u, \quad R(x) = g(x)\bar{R}g^T(x)$$
$$y = g^T(x)H_x^T(x)$$

Dissipation in port-controlled Hamiltonian system

For a system

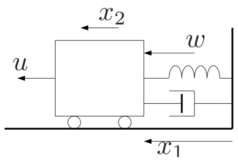
$$\dot{x} = (J(x) - R(x))H_x^T(x) + g(x)u$$
$$y = g^T(x)H_x^T(x)$$

one has the dissipation inequality

$$\int_0^T y^T u\, dt + H(x(0)) - H(x(T)) = \int_0^T H_x R H_x^T\, dt \geq 0$$

if R is positive semidefinite.

A mechanical example with nonlinear spring



x_1 position, x_2 momentum , m mass

$$H = \underbrace{\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4}_{H_1} + \underbrace{\frac{1}{2m}x_2^2}_{H_2}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{J(x)} - \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{R(x)} \right) \begin{bmatrix} x_1 + x_1^3 \\ x_2/m \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 + x_1^3 \\ x_2/m \end{bmatrix}$$

Some design ideas

Using the state feedback $u = k(x)$ where k satisfies

$$(J(x) - R(x))\bar{H}_x^T = g(x)k(x)$$

the new port controlled, damped Hamiltonian system

$$\dot{x} = (J(x) - R(x))(H + \bar{H})_x^T$$

is created, where the Hamiltonian is changed from H to $H + \bar{H}$. By using

$$(\bar{J}(x) - \bar{R}(x))H_x^T = g(x)k(x)$$

the interconnection and damping is changed instead:

$$\dot{x} = (J(x) + \bar{J}(x) - R(x) - \bar{R}(x))H_x^T$$

A reference

R. Ortega, A. J. van der Schaft, I Mareels, B. Maschke: Putting Energy Back in Control, IEEE Control Systems Magazine, April 2001.