Target Tracking
Le 6: RFS tracking

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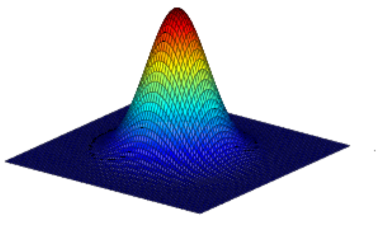
Bayesian Inference

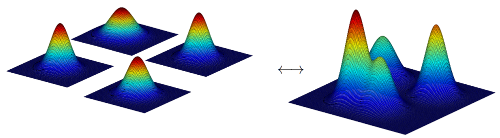


(Single) Target tracking

All information about the state is described by the *posterior*, given by Bayes' rule:

$$p(\boldsymbol{x}|\boldsymbol{z}) = \frac{g(\boldsymbol{z}|\boldsymbol{x})\, p(\boldsymbol{x})}{\int g(\boldsymbol{z}|\xi)\, p(\xi)\, d\xi}$$

Bayesian Multi-Object Inference



Multi-Target tracking

All information about the states is described by the *posterior*, given by Bayes' rule:

$$\pi(\mathcal{X}|\mathcal{Z}) = \frac{g(\mathcal{Z}|\mathcal{X})\, \pi(\mathcal{X})}{\int g(\mathcal{Z}|\Xi)\, p(\Xi)\, d\Xi}$$

where a *set integral* needs to be defined.

Random Finite Sets

Definition: Random Finite Set

A Random Finite Set (RFS) $\mathbf{X}$ is a random variable that has realizations in the form $\mathbf{X} = \mathcal{X} \in \mathcal{S}$ where $\mathcal{S}$ is the set of all finite subsets of some underlying space $\mathbb{S}$.

- The number of points is random.
- The points are random.
- The points have no ordering

RFS Example

- $\mathbb{S} = \mathbb{R}^{n_x}$
- $\mathcal{S} =$ All finite subsets of $\mathbb{R}^{n_x}$
- Let $x_k^i \in \mathbb{R}^{n_x}$ for $i = 1, \ldots, \infty$. Then, some realizations $\mathcal{X}$ of the random variable $\mathbf{X}$ can be ϕ, $\{\boldsymbol{x}_k^1\}$, $\{\boldsymbol{x}_k^1, \boldsymbol{x}_k^2\}$, $\{\boldsymbol{x}_k^1, \boldsymbol{x}_k^2, \boldsymbol{x}_k^3\}$ and so on.

Poisson Random Sets

Recall: Poisson Point Mass Function

$$Po\,(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Poisson RFS

Let $|\mathcal{X}|$ be the cardinality of $\mathcal{X}$ and $\langle f, g\rangle$ the inner product of functions f and g, such that $\langle f, g\rangle \triangleq \int f\,(\boldsymbol{x})\,g\,(\boldsymbol{x})\,dx$.

An RFS $\mathcal{X}$ is said to be Poisson with intensity function $v(\boldsymbol{x})$ if

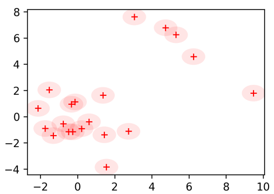
1. for $\mathcal{B} \subseteq \mathcal{X}$, $|\mathcal{X} \bigcap \mathcal{B}|$ is Poisson distributed with mean $\langle v, 1_{\mathcal{B}}\rangle$.
2. for any disjoint $\mathcal{B}_1, \ldots \mathcal{B}_i$, $|\mathcal{X} \bigcap \mathcal{B}_1|, \ldots, |\mathcal{X} \bigcap \mathcal{B}_i|$ are independent

Poisson Random Sets

The Probability Density Function (PDF) of a Poisson RFS is

$$\pi\,(\mathcal{X}) = e^{-\langle v,1\rangle} v^{\mathcal{X}}$$

- The distribution is characterized by the intensity function $v(\boldsymbol{x})$ (Mahler and Zajic, 2001; Mahler, 2003).
- If hyper-volume (on $\mathcal{X}$) has dimension κ, the intensity function $v(\boldsymbol{x})$ has the dimension κ^{-1}



Random Sets: Bayesian filter

- The specification of a Bayesian filter on sets is very similar to the standard Bayesian density recursion where related densities and operations are replaced with their set equivalents.

$$p(\mathcal{X}_k|\mathcal{Z}_{0:k}) \propto p(\mathcal{Z}_k|\mathcal{X}_k) \int p(\mathcal{X}_k|\mathcal{X}_{k-1}) p(\mathcal{X}_{k-1}|\mathcal{Z}_{0:k-1}) \delta\mathcal{X}_{k-1}$$

  where a **set integral** of a function f is necessary:
  $\int f\,(\mathcal{X})\,d\mathcal{X} = f\,(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f\,(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})\,d_{\boldsymbol{x}_1} \ldots d_{\boldsymbol{x}_n}$
- This general filter is computationally prohibitive to implement except few cases.

Random Sets: Models

- Call the target set at time k as $\mathcal{X}_k$ and measurement set at time k as $\mathcal{Z}_k$.
- Then, one can define set models

$$\mathcal{X}_k = F(\mathcal{X}_{k-1}) \cup W_k$$

where W_k the finite set representing the newly appearing targets. The function $F(\cdot)$ is related to target death and modelled prediction update of targets.

$$\mathcal{Z}_k = G(\mathcal{X}_k) \cup V_k$$

where V_k is the finite set representing the clutter. The function $G(\cdot)$ is related to the detection of the targets.

Approximative Models

Single-target moments

"Assuming Gaussian..."

$$\text{Mean: } \boldsymbol{\mu} = \int \boldsymbol{x} p(\boldsymbol{x}) d\boldsymbol{x}$$

$$\text{Covariance: } \boldsymbol{\Pi} = \int (\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^T p(\boldsymbol{x}) d\boldsymbol{x}$$

Kalman filter: $p(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Pi})$
Constant gain Kalman filter need only $\boldsymbol{x}$ (e.g. the α-β-γ-filter)

The mean, or the mean and covariance, describe an approximation of the true PDF

The Probability Hypothesis Density

What is the expected value (first moment) of a RFS?

- The multi-target moment is not straightforward (mean of sets is ill-defined)

~~$$\mathbb{E}[\mathcal{X}_k|\mathcal{Z}_{0:k}] = \int \mathcal{X}_k p(\mathcal{X}_k|\mathcal{Z}_{0:k}) \delta\mathcal{X}_k$$~~

- Needs an indirect first-order moment on the form

$$\mathbb{E}[h(\cdot)] = \int h(\mathcal{X})\, p(\mathcal{X}|\mathcal{Z})\, d\mathcal{X}$$

- Not, as one might expect, a clear set of "most likely tracks".
- The first moment of a random multitarget track-set is a density function, giving the expected number of targets at each (infinitesimal) point

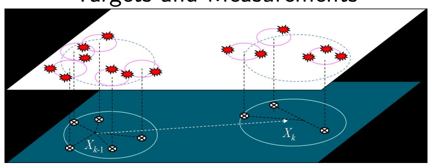
Probability Hypothesis Density (PHD)

- Suppose $\mathcal{X}_k = \{\boldsymbol{x}_k^1, \boldsymbol{x}_k^2, \ldots, \boldsymbol{x}_k^n\}$. Define a scalar valued function from $\mathcal{X}_k$ as follows that can be summed: $h_{\mathcal{X}_k}(\boldsymbol{x}) = \sum_{i=1}^n \delta_{\mathcal{X}_k^i}(\boldsymbol{x})$
- Then, the Probability Hypothesis Density (PHD) is the expectation of $h_{\mathcal{X}_k}(\boldsymbol{x})$ with respect to $\mathcal{X}_k$:

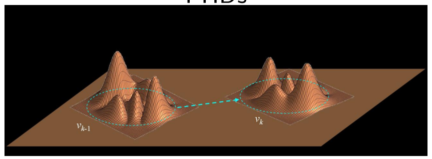
$$v_{k|k}(\boldsymbol{x}) = \mathbb{E}[h_{\mathcal{X}_k}(\boldsymbol{x})|\mathcal{Z}_{0:k}]$$

- $V(S) = \mathbb{E}[|\chi \cap S|] = \int_S v(\boldsymbol{x}) d\boldsymbol{x}$

Targets and Measurements



PHDs



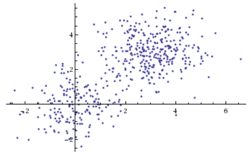
Figures obtained from random set filtering website
http://randomsets.eps.hw.ac.uk/

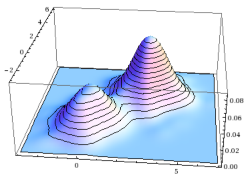
Probability Hypothesis Density (PHD) Filter

- PHD filter recursion

$$v_{k|k-1}(\boldsymbol{x}_k) = \int p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) v_{k-1|k-1}(\boldsymbol{x}_{k-1}) d\boldsymbol{x}_{k-1}$$

$$v_{k|k}(\boldsymbol{x}_k) = (1 - P_d(\boldsymbol{x}_k)) v_{k|k-1}(\boldsymbol{x}_k) + \sum_{\boldsymbol{z}_k \in \mathcal{Z}_k} \frac{P_d(\boldsymbol{x}_k) p(\boldsymbol{z}_k|\boldsymbol{x}_k) v_{k|k-1}(\boldsymbol{x}_k)}{\beta_{\mathsf{fa}} + \int P_d(\boldsymbol{x}_k) p(\boldsymbol{z}_k|\boldsymbol{x}_k) v_{k|k-1}(\boldsymbol{x}_k) d\boldsymbol{x}_k}$$





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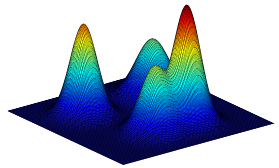
PHD Filter Implementation and Target Extraction

- Targets are identified from the peaks of the PHD.
- Using a PHD surface represented by a Gaussian mixture, the essential equations boils down to Kalman filter updates combined with a (*ad hoc*) mixture reduction step.
- E.g. a particle PHD representation requires additional peak extraction logic.



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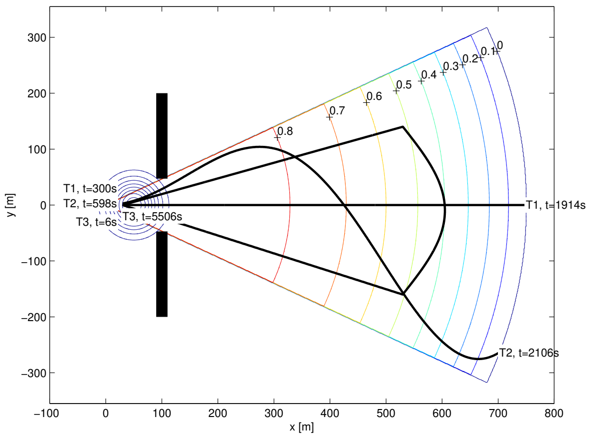
PHD Filter Example



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Probability Hypothesis Density (PHD) Filter

The PHD filter makes several assumptions (Vo et al., 2006)

- Targets evolve in time and generate independent measurements.
- The clutter RFS is Poisson and independent of the measurements.
- The predicted multitarget RFS is Poisson.

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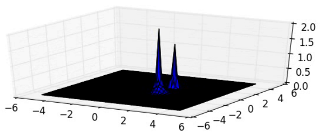
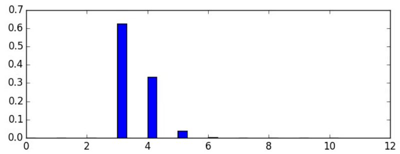
Cardinalized Probability Hypothesis Density (CPHD) Filter

(Mahler, 2006)

- The assumption of Poisson target cardinality makes the PHD sensitive to clutter. The Cardinalized Probability Hypothesis Density (CPHD) adds a full estimate of the cardinality distribution.
- "Spooky action at a distance"(Fränken et al., 2009): Missed measurements shifts the PHD from unrelated areas to detected parts.





Bernoulli Random Sets: Mathematical Definitions

Notation	Meaning
$1_{\mathcal{Y}}(\mathcal{X})$	This defines the inclusion function, such that $1_{\mathcal{Y}}(\mathcal{X}) \triangleq \begin{cases} 1, & \text{if } \mathcal{X} \subseteq Y, \\ 0, & \text{otherwise,} \end{cases}$
$\delta_{\mathcal{Y}}(\mathcal{X})$	Kronecker delta-function, used to select summands relevant to exactly the set $\mathcal{Y}$; $\delta_{\mathcal{Y}}(\mathcal{X}) \triangleq \begin{cases} 1, & \text{if } \mathcal{X} = \mathcal{Y}, \\ 0, & \text{otherwise,} \end{cases}$
$\mathcal{F}(\mathcal{X})$	All subsets of set $\mathcal{X}$
$h^{\mathcal{X}}$	Multi-object exponential notation, such that $h^{\mathcal{X}} \triangleq \prod_{\boldsymbol{x} \in \mathcal{X}} h(\boldsymbol{x})$ or $h^{\mathcal{X}} \triangleq \prod_{\boldsymbol{x} \in \mathcal{X}} h_{\boldsymbol{x}}$. $h^{\emptyset} = 1$ by convention

Bernoulli Random Sets

A Bernoulli RFS is a set with 0 or 1 elements according to a Bernoulli distribution with parameter r, i.e. for a set $\mathcal{X}$

- With probability $1 - r$, $\mathcal{X}$ is $\{\emptyset\}$
- With probability r, $\mathcal{X}$ is $\{\boldsymbol{x}\}$

If $\boldsymbol{x}$ is described by $p(\boldsymbol{x})$, the set is described with the Bernoulli RFS PDF

$$\pi(\chi) = \begin{cases} 1 - r, & \text{if } \chi = \emptyset, \\ r \cdot p(\boldsymbol{x}), & \text{if } \chi = \{\boldsymbol{x}\}. \end{cases}$$

Bernoulli RFS Parametrization

A Bernoulli RFS is fully described by the parameters

$$(r, p(\boldsymbol{x}))$$

Multi-Bernoulli Representation

A multi-Bernoulli RFS is the result of the union of N_{mb} independently Bernoulli-distributed RFS'S $\chi^{(i)}$, given by $\chi = \bigcup_{i=1}^{N_{mb}} \chi^{(i)}$.

- Multi-Bernoulli RFS

$$\pi(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}) = \prod_{j=1}^{M} (1 - r^{(j)}) \sum_{1 \le i_1 \ne \ldots, \ne i_n \le M} \prod \frac{r^{(i_j)} p^{(i_j)}(\boldsymbol{x}_j)}{1 - r^{(i_j)}}$$

$$\rho(n) = \prod_{j=1}^{M} (1 - r^{(j)}) \sum_{1 \le i_1 \ne \ldots, \ne i_n \le M} \prod \frac{r^{(i_j)}}{1 - r^{(i_j)}}$$

Multi-Bernoulli Random Sets

Multi-Bernoulli RFS Parametrization

The multi-Bernoulli RFS can be parametrized by the set

$$\left\{\left(r^{(i)}, p^{(i)}\right)\right\}_{i=1}^{N_{mb}}$$

Labeled Multi-Bernoulli RFS Parametrization

The Labeled multi-Bernoulli RFS can be parametrized by the set

$$\left\{\left(r^{(\ell)}, p^{(\ell)}\left(\boldsymbol{x}\right)\right)\right\}_{\ell\in\mathcal{L}}$$

δ-Generalized Labeled Multi-Bernoulli Filter

The δ-Generalized Labeled Multi-Bernoulli (δ-GLMB) PDF (Vo and Vo, 2013; Vo et al., 2014):

$$\pi\left(\mathcal{X}\right) = \Delta\left(\mathcal{X}\right) \sum_{(I,\xi)\in\mathcal{F}(\mathbb{L})\times\Xi} w^{(I,\xi)} \delta_I\left(\mathcal{L}\left(\mathcal{X}\right)\right)\left[p^{(\xi)}\right]^{\mathcal{X}}$$

- $\Delta\left(\mathcal{X}\right)$ is the distinct label indicator
$$\Delta\left(\mathcal{X}\right) = \delta_{|\mathcal{X}|}\left(\left|\mathcal{L}\left(\mathcal{X}\right)\right|\right)$$
- (I, ξ) loops over all hypotheses
- $w^{(I,\xi)}$ is the weight of the hypothesis
- $\delta_I\left(\mathcal{L}\left(\mathcal{X}\right)\right) \in \{0, 1\}$ filters out the hypotheses where exactly $\mathcal{X}$ exists.
- $\left[p^{(\xi)}\right]^{\mathcal{X}}$ are the PDFs of the tracks of the targets in $\mathcal{X}$.

δ-GLMB prediction update

Given a filtered δ-GLMB density, the predicted δ-GLMB is given by

$$\begin{aligned}
\pi_+\left(\mathcal{X}_+\right) &= \Delta\left(\mathcal{X}_+\right) \sum_{(I_+,\xi)\in\mathcal{F}(\mathbb{L}_+)\times\Xi} w_+^{(I_+,\xi)} \delta_{I_+}\left(\mathcal{L}\left(\mathcal{X}_+\right)\right)\left[p_+^{(\xi)}\right]^{\mathcal{X}_+} \\
w_+^{(I_+,\xi)} &= w_S^{(\xi)}\left(I_+ \bigcap \mathbb{L}\right) w_B\left(I_+ \bigcap \mathbb{B}\right) \\
w_S^{(\xi)}\left(L\right) &= \left[\eta_S^{(\xi)}\right]^L \sum_{I\supseteq L}\left[1-\eta_S^{(\xi)}\right]^{I-L} w^{(I,\xi)} \\
\eta_S^{(\xi)}\left(\ell\right) &= \left\langle p_S\left(\cdot,\ell\right), p^{(\xi)}\left(\cdot,\ell\right)\right\rangle \\
p_+^{(\xi)}\left(\boldsymbol{x},\ell\right) &= 1_{\mathbb{L}}\left(\ell\right) p_S^{(\xi)}\left(\boldsymbol{x},\ell\right) + 1_{\mathbb{B}} p_B\left(\boldsymbol{x},\ell\right) \\
p_S^{(\xi)}\left(\boldsymbol{x},\ell\right) &= \frac{\left\langle p_S\left(\cdot,\ell\right) f\left(\boldsymbol{x}|\cdot,\ell\right), p^{(\xi)}\left(\cdot,\ell\right)\right\rangle}{\eta_S^{(\xi)}\left(\ell\right)}
\end{aligned}$$

δ-GLMB measurement update

Given a predicted δ-GLMB, the posterior filtering density is given, with $\Theta\left(I\right)$ denoting the subset of current association maps with domain I, by:

$$\begin{aligned}
\pi\left(\mathcal{X}\right) &= \Delta\left(\mathcal{X}\right) \sum_{(I,\xi)\in\mathcal{F}(\mathbb{L})\times\Xi} \sum_{\theta\in\Theta(I)} w^{(I,\xi,\theta)} \delta_I\left(\mathcal{L}\left(\mathcal{X}\right)\right)\left[p^{(\xi,\theta)}\right]^{\mathcal{X}} \\
w^{(I,\xi,\theta)}\left(\mathcal{Z}\right) &\propto w^{(I,\xi,\theta)}\left[\eta_{\mathcal{Z}}^{(\xi,\theta)}\right]^I \\
\psi_{\mathcal{Z}}\left(\boldsymbol{x},\ell;\theta\right) &= \begin{cases} \frac{p_D(\boldsymbol{x},\ell) g\left(\boldsymbol{z}_{\theta(\ell)}|\boldsymbol{x},\ell\right)}{\kappa\left(\boldsymbol{z}_{\theta(\ell)}\right)}, & \theta\left(\ell\right) > 0, \\ 1 - p_D\left(\boldsymbol{x},\ell\right), & \theta\left(\ell\right) = 0, \end{cases} \\
\eta_{\mathcal{Z}}^{(\xi,\theta)}\left(\ell\right) &= \left\langle p^{(\ell)}, \psi_{\mathcal{Z}}\left(\cdot,\ell;\theta\right)\right\rangle \\
p^{(\xi,\theta)}\left(\boldsymbol{x},\ell|\mathcal{Z}\right) &= \frac{p^{(\xi)}\left(\boldsymbol{x},\ell\right)\psi_{\mathcal{Z}}\left(\boldsymbol{x},\ell;\theta\right)}{\eta_{\mathcal{Z}}^{(\xi,\theta)}\left(\ell\right)}
\end{aligned}$$

Linear Assignment Problem (LAP)

Linear Assignment Problem

The problem can be formulated by defining a cost matrix $\boldsymbol{C} \in \mathbb{R}^{n \times m}$, with matrix elements c_{ij} from row $i \in [1, \dots, n]$ and column $j \in [1, \dots, m]$:

$$
\begin{aligned}
\min \sum_{i,j} c_{ij} s_{ij} \\
\sum_j s_{ij} = 1, \quad \forall i, \quad \sum_i s_{ij} \leq 1, \quad \forall j \qquad (1) \\
s_{ij} \in \{0, 1\}
\end{aligned}
$$

LAP Example

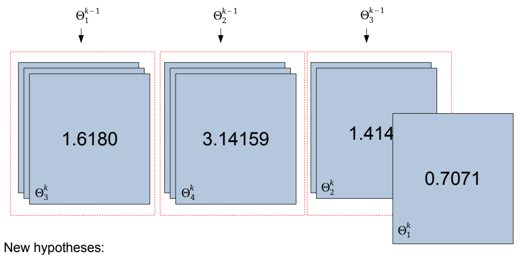
Assigning targets to reports

Given reports $\{\boldsymbol{z}_1, \boldsymbol{z}_2\}$ and targets $\{\ell_1, \ell_2\}$ we define the $\boldsymbol{C}$ matrix as

$$
\boldsymbol{C} = \begin{pmatrix} {}^{\boldsymbol{z}_1}\Lambda_{\ell_1} & {}^{\boldsymbol{z}_2}\Lambda_{\ell_1} & {}^{n}\Lambda_{\ell_1} & \infty & {}^{F}\Lambda_{\ell_1} & \infty \\ {}^{\boldsymbol{z}_1}\Lambda_{\ell_2} & {}^{\boldsymbol{z}_2}\Lambda_{\ell_2} & \infty & {}^{n}\Lambda_{\ell_2} & \infty & {}^{F}\Lambda_{\ell_2} \end{pmatrix},
$$

where ${}^{\boldsymbol{z}_j}\Lambda_{\ell_i}$ is the cost assigned to associating target ℓ_j to report $\boldsymbol{z}_j$. ${}^{n}\Lambda_{\ell_i}$ and ${}^{F}\Lambda_{\ell_j}$ is the cost associated with assigning the target as non-associated or false, respectively.

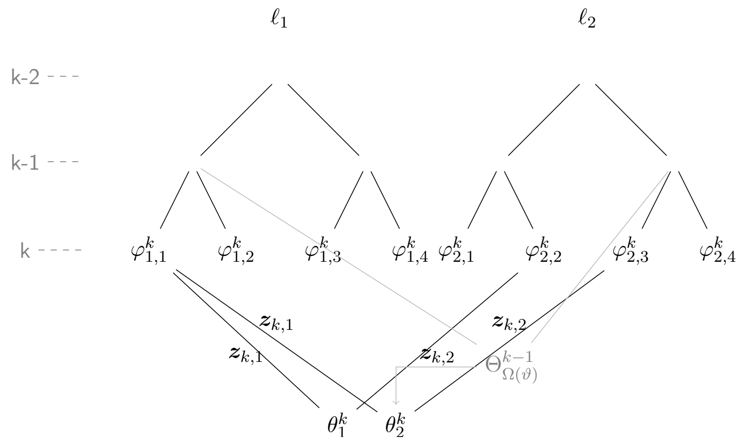
Building upon old hypothetical tracks



New hypotheses:

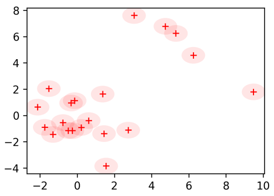
Order	Score	Name	Parent
1	0.7071	Θ_1^k	Θ_3^{k-1}
2	1.414	Θ_2^k	Θ_3^{k-1}
3	1.6180	Θ_3^k	Θ_1^{k-1}
4	3.14159	Θ_4^k	Θ_2^{k-1}

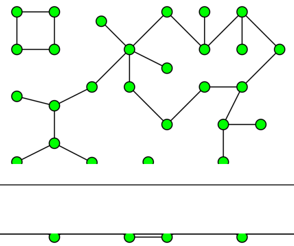
Hypothesis Generation



Clustering

- δ-GLMB, like MHT is difficult to cluster
  - Reports that are, in even just a single hypothesis, associated to separate targets will connect those targets to the same cluster.
  - Targets that have ever been connected must stay in the same cluster "forever" (until ambiguity has been resolved or discarded).





Labeled Multi-Bernoulli Filter

- A δ-GLMB representation with only a single hypothesis is a Labeled Multi-Bernoulli (LMB)
- The δ-GLMB measurement update of a LMB PDF is a δ-GLMB
- LMB filter idea: Approximate the δ-GLMB with a LMB
- Recall: LMB PDF is parametrized by

$$\left\{\left(r^{(\ell)}, p^{(\ell)}(\boldsymbol{x})\right)\right\}_{\ell \in \mathcal{L}}$$

Labeled Multi-Bernoulli Filter: Prediction

Chapman-Kolmogorov equation:

$$\pi_+(\mathcal{X}_+) = \int f(\mathcal{X}_+)\,\pi(\mathcal{X})\,\delta\mathcal{X},$$

This gives the following set of surviving and new-born targets (Reuter et al., 2014),

$$\pi_+ = \left\{\left(r_{+,s}^{(\ell)}, p_{+,s}^{(\ell)}\right)\right\}_{\ell \in \mathcal{L}} \cup \left\{\left(r_B^{(\ell)}, p_B^{(\ell)}\right)\right\}_{\ell \in \mathcal{B}}$$

where

$$r_{+,s}^{(\ell)} = \eta_s(\ell)\, r^{(\ell)},$$

$$p_{+,s}^{(\ell)} = \frac{\langle p_s(\cdot,\ell)\, f(\boldsymbol{x}|\cdot,\ell),\, p(\cdot|\ell)\rangle}{\eta_s(\ell)},$$

$$\eta_s(\ell) = \langle p_s(\cdot,\ell),\, p(\cdot,\ell)\rangle,$$

RFS Birth Models

- We wish to form

$$\pi_{B,k+1} = \left\{\left(r_B^{(\ell)}, p_B^{(\ell)}\right)\right\}_{\ell \in \mathcal{B}_k}$$

- One (ad hoc) model is based on the probability of association:

$$r_{U,k}(\boldsymbol{z}) = \sum_{(\mathcal{I}_+,\theta)\in\mathcal{F}\left(\mathcal{L}_+^{(\zeta)}\right)\times\Theta_{\mathcal{I}_+}} w^{(\mathcal{I}_+,\theta)}\left(\mathcal{Z}^{(\zeta)}\right) 1_\theta(\boldsymbol{z}).$$

Given an expected number of new targets in each scan, $\lambda_{B,k+1}$ — the existence probability of new targets — is then given by

$$r_{B,k+1}(\boldsymbol{z}) = \min\left(r_B^{\max}, \frac{\left(1 - r_{U,k}(\boldsymbol{z})\right)\cdot\lambda_{B,k+1}}{\sum_{\boldsymbol{z}'\in\mathcal{Z}_k} 1 - r_{U,k}(\boldsymbol{z}')}\right).$$

Labeled Multi-Bernoulli Filter: Measurement update

The measurement updates the set $\pi_+ = \left\{\left(r_+^{(\ell)}, p_+^{(\ell)}\right)\right\}_{\ell\in\mathcal{L}_+}$ by the following approximation, for N_ζ clusters:

$$\pi\left(\cdot|\mathcal{Z}\right) \approx \left\{\left(r^{(\ell)}, p^{(\ell)}\right)\right\}_{\ell\in\mathcal{L}_+} = \bigcup_{\zeta=1}^{N_\zeta} \left\{\left(r^{(\ell,\zeta)}, p^{(\ell,\zeta)}\right)\right\}_{\ell\in\mathcal{L}_+^{(\zeta)}},$$

in which parameters are given by

$$r^{(\ell,\zeta)} = \sum_{(\mathcal{I}_+,\theta)\in\mathcal{F}\left(\mathcal{L}_+^{(\zeta)}\right)\times\Theta_{\mathcal{I}_+}} w^{(\mathcal{I}_+,\theta)}\left(\mathcal{Z}^{(\zeta)}\right) 1_{\mathcal{I}_+}\left(\ell\right),$$

$$p^{(\ell,\zeta)}\left(\boldsymbol{x}\right) = \frac{1}{r^{(\ell,\zeta)}} \sum_{(\mathcal{I}_+,\theta)\in\mathcal{F}\left(\mathcal{L}_+^{(\zeta)}\right)\times\Theta_{\mathcal{I}_+}} w^{(\mathcal{I}_+,\theta)}\left(\mathcal{Z}^{(\zeta)}\right) \times 1_{\mathcal{I}_+}\left(\ell\right) p^{(\theta)}\left(\boldsymbol{x},\ell|\mathcal{Z}^{(\zeta)}\right)$$

Labeled Multi-Bernoulli Filter: Measurement update

$$w^{(\mathcal{I}_+,\theta)}\left(\mathcal{Z}^{(\zeta)}\right) \propto w_{+,\zeta}^{(\mathcal{I}_+)}\left[\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta)}\right]^{\mathcal{I}_+}$$

$$w_{+,\zeta}^{(\mathcal{I}_+)} = \prod_{\ell\in\mathcal{L}_+^{(\zeta)}-\mathcal{I}_+}\left(1-r_+^{(\ell)}\right)\prod_{\ell'\in\mathcal{I}_+} r_+^{(\ell)},$$

$$\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta)}\left(\ell\right) = \left\langle p_+^{(\ell,\zeta)}\left(\boldsymbol{x}\right), \psi_{\mathcal{Z}^{(\zeta)}}\left(\cdot,\ell;\theta\right)\right\rangle$$

$$\psi_{\mathcal{Z}^{(\zeta)}}\left(\boldsymbol{x},\ell;\theta\right) = \begin{cases} \frac{p_D(\boldsymbol{x},\ell)p_G g\left(\boldsymbol{z}_{\theta(\ell)}|\boldsymbol{x},\ell\right)}{\kappa\left(\boldsymbol{z}_{\theta(\ell)}\right)}, & \theta\left(\ell\right) \neq \boldsymbol{z}_\emptyset, \\ q_{D,G}\left(\boldsymbol{x},\ell\right), & \theta\left(\ell\right) = \boldsymbol{z}_\emptyset, \end{cases}$$

$$q_{D,G}\left(\boldsymbol{x},\ell\right) = 1 - p_D\left(\boldsymbol{x},\ell\right)p_G,$$

$$p^{(\theta)}\left(\boldsymbol{x},\ell|\mathcal{Z}^{(\zeta)}\right) = \frac{p_+^{(\ell,\zeta)}\left(\boldsymbol{x}\right)\psi_{\mathcal{Z}^{(\zeta)}}\left(\boldsymbol{x},\ell;\theta\right)}{\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta)}\left(\ell\right)}$$

LMB Implementation

We make the distinction between associated and non-associated targets:

$$\mathcal{I}_+^a = \{\ell : \quad \theta\left(\ell\right) \neq \boldsymbol{z}_\emptyset\}_{\ell\in\mathcal{I}_+},$$

$$\mathcal{I}_+^n = \{\ell : \quad \theta\left(\ell\right) = \boldsymbol{z}_\emptyset\}_{\ell\in\mathcal{I}_+},$$

(implying $\mathcal{I}_+ = \mathcal{I}_+^a \cup \mathcal{I}_+^n$ and $\mathcal{I}_+^a \cap \mathcal{I}_+^n = \emptyset$). We can then rewrite the measurement update

$$\begin{aligned} w^{(\mathcal{I}_+,\theta)}\left(\mathcal{Z}^{(\zeta)}\right) &\propto w_{+,\zeta}^{(\mathcal{I}_+)}\left[\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta)}\right]^{\mathcal{I}_+} \\ &= \prod_{\ell\in\mathcal{L}_+^{(\zeta)}-\mathcal{I}_+}\left(1-r_+^{(\ell)}\right) \\ &\times \prod_{\ell'\in\mathcal{I}_+^a} r_+^{(\ell')}\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta)}\left(\ell'\right) \prod_{\ell''\in\mathcal{I}_+^n} r_+^{(\ell'')}\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta)}\left(\ell''\right), \end{aligned}$$

LMB Implementation

This product can be efficiently expressed using the Negative Log Likelihoods (NLLs), Λ_ℓ;

$$e^{-\Lambda_\ell} = \begin{cases} 1 - r_+^{(\ell)}, & \text{if } \ell \in \mathcal{L}_+^{(\zeta)} - \mathcal{I}_+, \\ r_+^{(\ell)}\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta,a)}\left(\ell\right), & \text{if } \ell \in \mathcal{I}_+^a, \\ r_+^{(\ell)}\eta_{\mathcal{Z}^{(\zeta)}}^{(\theta,n)}\left(\ell\right), & \text{if } \ell \in \mathcal{I}_+^n, \end{cases}$$

yielding

$$w^{(\mathcal{I}_+,\theta)}\left(\mathcal{Z}^{(\zeta)}\right) \propto \exp\left(-\sum_{\ell\in\mathcal{L}_+^{(\zeta)}}\Lambda_\ell\right).$$

LAP Recap

- LAP formulation:

$$\begin{aligned} &\min \sum_{i,j} c_{ij} s_{ij} \\ &\sum_j s_{ij} = 1, \quad \forall i, \quad \sum_i s_{ij} \leq 1, \quad \forall j \\ &s_{ij} \in \{0,1\} \end{aligned} \tag{2}$$

$$\boldsymbol{C} = \begin{pmatrix} {}^{z_1}\Lambda_{\ell_1} & {}^{z_2}\Lambda_{\ell_1} & {}^{n}\Lambda_{\ell_1} & \infty & {}^{F}\Lambda_{\ell_1} & \infty \\ {}^{z_1}\Lambda_{\ell_2} & {}^{z_2}\Lambda_{\ell_2} & \infty & {}^{n}\Lambda_{\ell_2} & \infty & {}^{F}\Lambda_{\ell_2} \end{pmatrix},$$

- Each hypothesis describes a combination of elements which can be summed!

LMB Measurement Update Reformulation

- Add a virtual "missed" measurement to the set of measurements

$$\mathcal{Z}^{\dagger} = \mathcal{Z} \cup \{\boldsymbol{z}_{\emptyset}\}.$$

- Then $p^{(\theta)}(\boldsymbol{x}, \ell | \mathcal{Z})$ belongs to a limited set:

$$p^{(\theta)}(\boldsymbol{x}, \ell | \mathcal{Z}) \in \left\{ p^{(\ell)}(\boldsymbol{x}|\boldsymbol{z}) \right\}_{\boldsymbol{z} \in \mathcal{Z}^{\dagger}}.$$

- We define the assignment indicator function

$$A^{\Theta}_{\boldsymbol{z} \leftrightarrow \ell} \triangleq \begin{cases} 1, & \text{if } \Theta \text{ assigns label } \ell \text{ to report } \boldsymbol{z}, \\ 0, & \text{otherwise.} \end{cases}$$

LMB Measurement Update Reformulation

- Abbreviating $w^{\theta} = w^{(\mathcal{I}_+, \theta)}\left(\mathcal{Z}^{(\zeta)}\right)$ and denoting the inner sums as ${}^{z}w_{\ell}$:

$$\begin{aligned} r^{(\ell)} &= \sum_{\boldsymbol{z} \in \mathcal{Z}^{\dagger}} \left[\sum_{(\mathcal{I}_+, \theta) \in \mathcal{F}(\mathcal{L}_+) \times \Theta_{\mathcal{I}_+}} w^{\theta} A^{\theta}_{\boldsymbol{z} \leftrightarrow \ell} \right] \\ &= \sum_{\boldsymbol{z} \in \mathcal{Z}^{\dagger}} {}^{z}w_{\ell} \\ p^{(\ell)}(\boldsymbol{x}) &= \frac{1}{r^{(\ell)}} \sum_{\boldsymbol{z} \in \mathcal{Z}^{\dagger}} \left[\sum_{(\mathcal{I}_+, \theta) \in \mathcal{F}(\mathcal{L}_+) \times \Theta_{\mathcal{I}_+}} w^{\theta} A^{\theta}_{\boldsymbol{z} \leftrightarrow \ell} \right] p^{(\ell)}(\boldsymbol{x}|\boldsymbol{z}) \\ &= \frac{1}{r^{(\ell)}} \sum_{\boldsymbol{z} \in \mathcal{Z}^{\dagger}} {}^{z}w_{\ell} p^{(\ell)}(\boldsymbol{x}|\boldsymbol{z}) \end{aligned}$$

We see that ${}^{z}w_{\ell}$ is *the sum of weights of all hypotheses that assign report $\boldsymbol{z}$ to label ℓ.*

Birth Model Reformulation

Further, the birth model of may be rewritten

$$\begin{aligned} r_{U,k}(\boldsymbol{z}) &= \sum_{\ell \in \mathcal{L}_+^{(\zeta)}} \left[\sum_{(\mathcal{I}_+, \theta) \in \mathcal{F}\left(\mathcal{L}_+^{(\zeta)}\right) \times \Theta_{\mathcal{I}_+}} w^{\theta} A^{\theta}_{\boldsymbol{z} \leftrightarrow \ell} \right] && (3) \\ &= \sum_{\ell \in \mathcal{L}_+^{(\zeta)}} {}^{z}w_{\ell}. && (4) \end{aligned}$$

LMB Implementation: Efficient Algorithm

To exploit this reformulation, consider a cluster of $N_\mathcal{X}$ targets and $N_\mathcal{Z}$ reports, and a matrix $\boldsymbol{W} \in \mathbb{R}^{N_\mathcal{X} \times (N_\mathcal{Z}+2)}$. Further, consider a hypothesis assignment mapping $R_\theta(i)$ to be used for mapping each row index of $\boldsymbol{W}$ (corresponding to a target) to a column index (corresponding to an assignment).

Assignment mapping

For all known targets (rows), $R_\theta(i)$

1. maps associated targets to its report's integer position in an ordered enumeration of the reports;
2. maps missed targets to the integer index $N_\mathcal{Z} + 1$; and
3. maps false targets to the integer index $N_\mathcal{Z} + 2$.

LMB Implementation: Efficient Algorithm

See (Olofsson, 2019)

Algorithm 1 Weight matrix calculation

$\boldsymbol{W} \leftarrow N_\mathcal{X} \times (N_\mathcal{Z}+2)$ zero matrix.
$s \leftarrow 0$
for $(w^\theta, \theta) \in \text{murty}(\boldsymbol{C})$ **do**
  $\boldsymbol{W}[i, R_\theta(i)] \leftarrow \boldsymbol{W}[i, R_\theta(i)] + w^\theta, \quad \forall i \in [1, \dots, N_\mathcal{X}]$
  $s \leftarrow s + w^\theta$
end for
$\boldsymbol{W} \leftarrow \frac{\boldsymbol{W}}{s}$

$r^{(\ell)}$

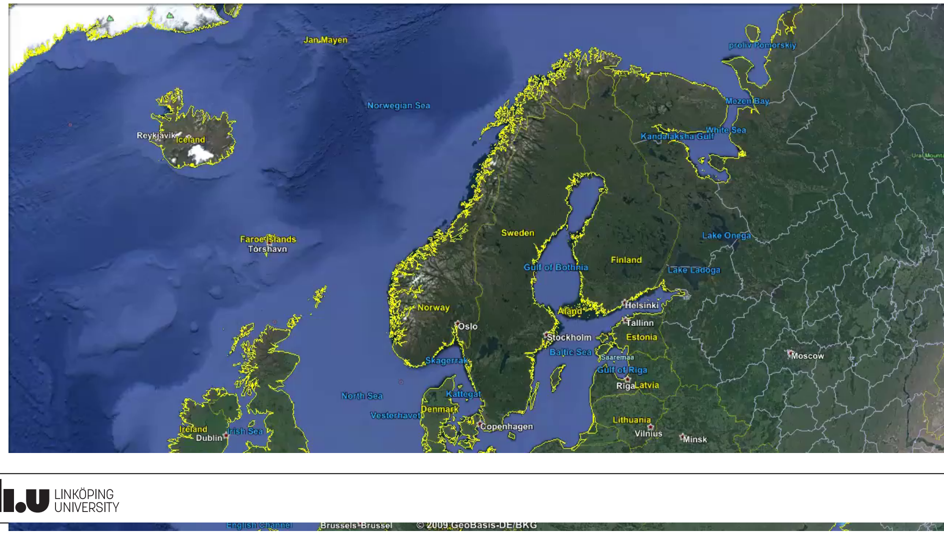
$$\boldsymbol{W} = \begin{pmatrix} {}^{z_1}w_{\ell_1} & {}^{z_2}w_{\ell_1} & {}^{\emptyset}w_{\ell_1} & {}^{F}w_{\ell_1} \\ {}^{z_1}w_{\ell_2} & {}^{z_2}w_{\ell_2} & {}^{\emptyset}w_{\ell_2} & {}^{F}w_{\ell_2} \\ {}^{z_1}w_{\ell_3} & {}^{z_2}w_{\ell_3} & {}^{\emptyset}w_{\ell_3} & {}^{F}w_{\ell_3} \end{pmatrix}$$

$r_{U,k}(z)$

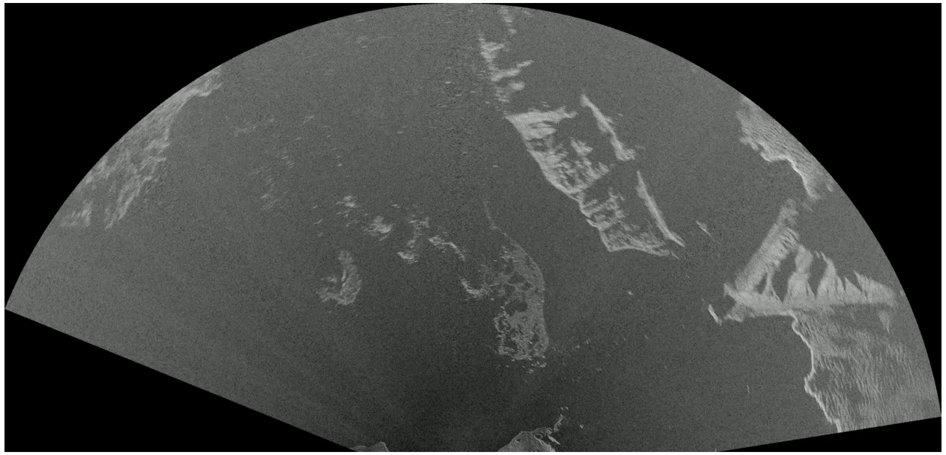
LMB MU Implementation

```
Assignment res;
double cost;
double w;
double w_sum = 0;
Eigen::MatrixXd R(M, N + 1);
R.setZero();
// Draw most relevant hypotheses using Murty's algorithm
while (murty.draw(res, cost)) {
    w = std::exp(-cost);
    w_sum += w;
    for (unsigned i = 0; i < M; ++i) {
        if ((unsigned)res[i] < N) {
            R(i, res[i]) += w;
        } else if ((unsigned)res[i] == N + i) {
            R(i, N) += w;
        }
    }
    ++n;
    if (w / w_sum < params->w_lim || n >= params->nhyp_max) {
        break;
    }
}
```

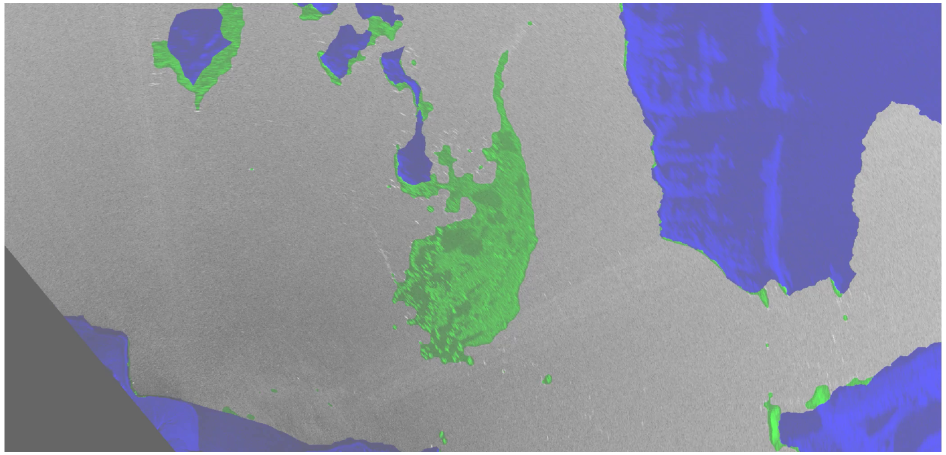

Sea Ice Monitoring with LMB

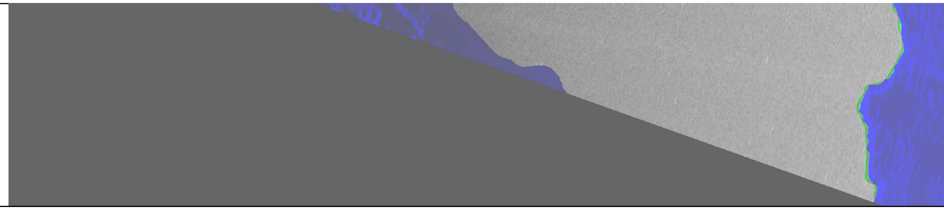


Sea Ice Monitoring with LMB



Sea Ice Monitoring with LMB





Cutting Edge: Poisson Multi-Bernoulli Mixture (PMBM)

- Explicitly models yet undetected targets as an integrated Poisson distributed set.
- Can be implemented as a version of MHT (like the δ-GLMB filter).

See e.g. (García-Fernández et al., 2018)

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