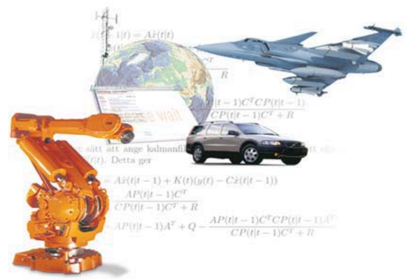
Lecture 1 – Rigid Body Motion



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Content

- Rigid body transformation
- Rotation
  - Rotation matrices
  - Euler's theorem
  - Parameterization of SO(3)
- Homogeneous representation
  - Matrix representation
  - Chasles' theorem

Background to modeling

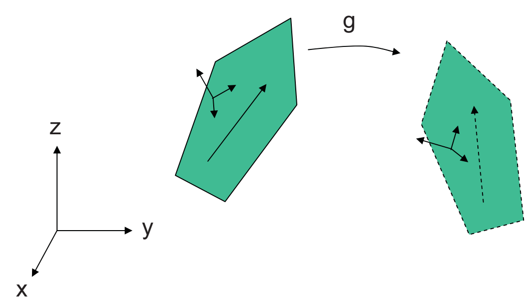
Kinematics

- studies the motion of objects without consideration of the circumstances leading to the motion

Dynamics

- studies the relationship between the motion of objects and its causes

Rigid body motion

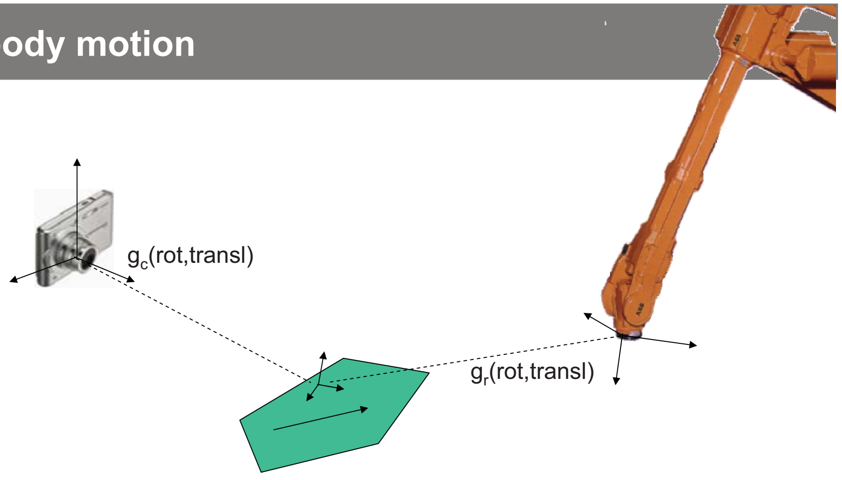


The motion of a rigid body can be parameterized as

- position
- orientation

of one point of the object. The ***configuration***.

Rigid body motion



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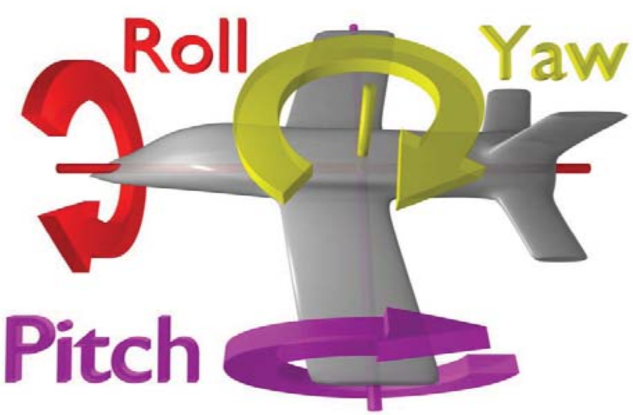
Representation of orientation

- Angle – axis representation
- Euler angles
- Quaternion
- Exponential coordinates
- …

Euler angles



Euler angles

☹ The order of rotation axes is important

i) *ii)* *iii)*

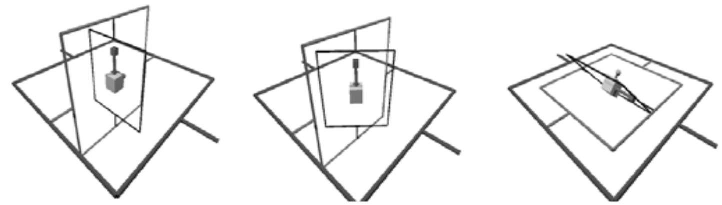
$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

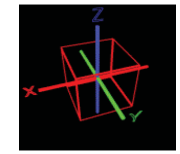
$$R_2(\alpha) = \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}$$

$$R_3(\alpha) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler angles

☹ Gimbal lock (Apollo IMU Gimbal lock 1, 2)





$$R(\alpha, \tfrac{\pi}{2}, \gamma) = \begin{pmatrix} 0 & \cos\gamma\sin\alpha - \cos\alpha\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & 0 \\ 0 & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\gamma - \cos\gamma\sin\alpha & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \\ 0 & \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Euler angles

☹ Implementing interpolation is difficult
☹ Ambiguous correspondence to rotations
☹ The result of composition is not apparent
☹ Non-linear dynamics

☺ Mathematics is well known
☺ Can be visualized "in the mind"

Quaternions

Sir William Rowan Hamilton (1809-1865)

Lectures on Quaternions: Containing a systematic statement of

𝔄 𝔑𝔢𝔴 𝔐𝔞𝔱𝔥𝔢𝔪𝔞𝔱𝔦𝔠𝔞𝔩 𝔐𝔢𝔱𝔥𝔬𝔡

of which the principles were communicated in 1843 to the Royal Irish Academy; and which has since formed the subject of successive courses of lectures, delivered in 1848 and subsequent years in the halls of Trinity College, Dublin: with numerous illustrative diagrams, and with some geometrical and physical applications.

Quaternions

Generalization of complex numbers to 3D.

s + i x+j y+k z

with $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

A quaternion is usually represented as $q = <s,v>$ with

- s scalar (real part)
- v vector in R^3 (complex part)

Unit quaternion $\|q\| = 1$.

Rotation with quaternions

Angle axis to quaternion

$$\theta, v \Rightarrow \quad q = \left\langle \cos\frac{\theta}{2}, \sin\frac{\theta}{2} v \right\rangle$$

Composition of rotations, q_1 then q_2

$$q = q_2 q_1$$

Rotation with quaternions

Rotation of a vector, $u = Rv$

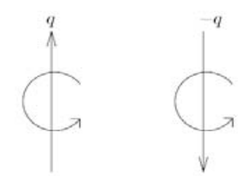
$v_q = <0, v>$, q is quaternion representation of R

$$u_q = q v_q q^{-1} = <0, u>$$

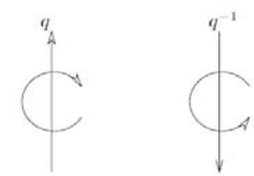
$$R_q(\mathbf{q}) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.$$

Some remarks

- q and $-q$ represent the same rotation



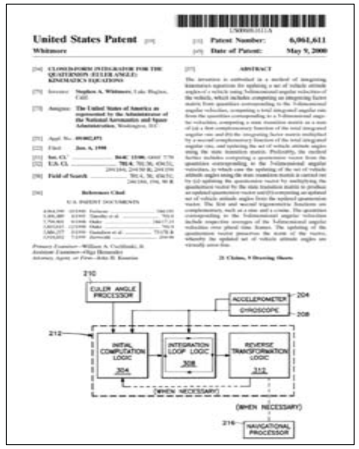
- $q = <s,v>$ and $q^{-1} = <s,-v>$



Quaternions

- ☹ Can only represent orientation
- ☹ Quaternion math is not so well known

- ☺ Compact representation, based upon angle axis rep.
- ☺ Simple interpolation methods
- ☺ No gimbal lock
- ☺ Simple composition
- ☺ Linear (bi-linear) dynamics, (NASA)



Comparison for different operations

Performance comparison of rotation chaining operations

Method	Storage	# multiplies	# add/subtracts	total operations
Rotation matrix	9	27	18	45
Quaternions	4	16	12	28

Performance comparison of various rotation operations

Method	Storage	# multiplies	# add/subtracts	# sin/cos	total operations
Rotation matrix	9	9	6	0	15
Quaternions	4	21	18	0	39
Angle/axis	4*	23	16	2	41

Illustration of Euler's Theorem

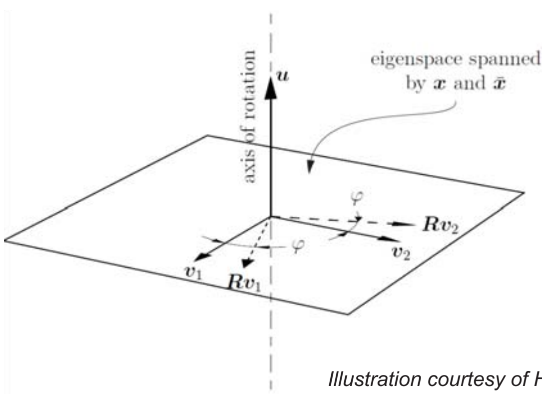


Illustration courtesy of Henrik Tidefelt, Shapes.

Hence, the effect of the rotation R is to rotate vectors in the plane spanned by v_1 and v_2 through an angle φ along u. This shows that that R rotates a rigid body about u through an angle φ. This concludes the proof of Euler's theorem.

Canonical Representation of the Rotation Matrix

There is a canonical representation of any rotation matrix R, that allows us to view it as a rotation through an angle φ about the z-axis.

Define the orthonormal matrix $Q = \begin{pmatrix} v_1 & | & v_2 & | & u \end{pmatrix}$

and

$$\Lambda = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

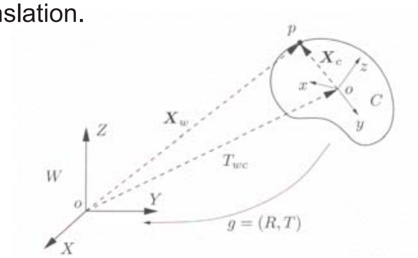
Then we can show that

$$R = Q\Lambda Q^T$$

Recall that *"change of basis = similarity transformation"*

Homogeneous Representation

How do we represent rigid body motion in general, i.e., both orientation and translation.



$$X^w = R^{wc} X^c + T^{wc}$$

A full rigid-body motion is denoted by $g = (R, T)$

The set of all possible configurations of a rigid body can be described by the space of rigid-body motions or **special Euclidean** transformations

$$SE(3) \triangleq \left\{ g = (R, T) | R \in SO(3), T \in \mathbb{R}^3 \right\}$$

Homogeneous Representation

The equation

$$X^w = R^{wc} X^c + T^{wc}$$

is affine, we would like to get rid of the additive term.

We can convert the affine transformation into a linear transformation by augmenting an additional 1 to X

$$\bar{X} = \begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{pmatrix}$$

Homogeneous Representation

What is linear about this?

Let us have a look

$$\bar{X}^w = \begin{pmatrix} X^w \\ 1 \end{pmatrix} = \begin{pmatrix} R^{wc} & T^{wc} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^c \\ 1 \end{pmatrix} = \bar{g}^{wc} \bar{X}^c$$

This leads us to the so called homogeneous representation of the special Euclidean transformations

$$SE(3) \triangleq \left\{ \bar{g} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \middle| R \in SO(3), T \in \mathbb{R}^3 \right\}$$

Chasle's Theorem

Proof:

Consider a general 4x4 homogeneous matrix (describing a rigid body motion)

$$A = \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix}$$

We will now change basis in order to see better (again, recall that "change of basis = similarity transform").

Perform a similarity transformation of the matrix A

$$\begin{aligned} \Lambda &= \begin{pmatrix} Q^T & -Q^T c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & c \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} Q^T R Q & Q^T R c - Q^T c + Q^T d \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Chasle's Theorem

Proof (continued):

Rotation:

Recall that v_1, v_2 and u are orthogonal

Choose Q according to $Q = \begin{pmatrix} v_1 & | & v_2 & | & u \end{pmatrix}$ $\Longrightarrow$

$$Q^T R Q = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a rotation about the z-axis

Chasle's Theorem

Hence, the rigid body motion is described by a rotation about the z-axis through an angle φ followed by a translation along the z-axis through a distance k.

If the top 2x2 submatrix of (Q^TRQ - I) is singular, then Q^TRQ = I. This means that Λ is a pure translation.

The proof is finished.

Chasle's Theorem

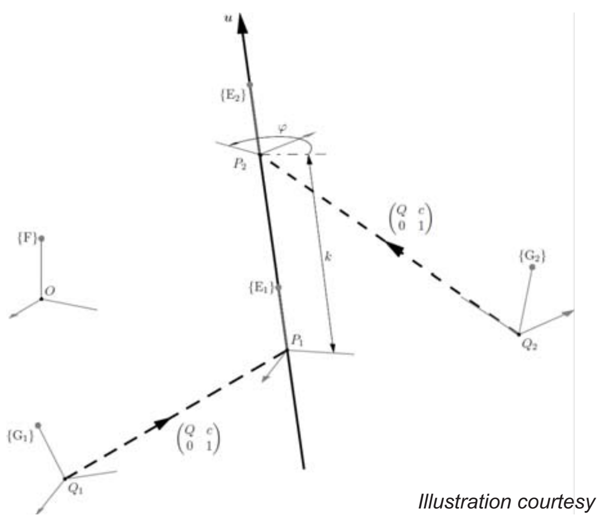
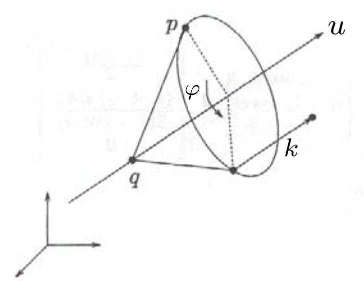


Illustration courtesy of Henrik Tidefelt, Shapes

Chasle's Theorem – Screw Motion

The motion implied by Chasle's theorem is like when you screw in that it rotates and translates along the same axis.



Further Studies Besides Course Litterature

- R.M. Murray, Z. Li, and S.S. Sastry: **A mathematical introduction to Robotic Manipulation** (Chapter 2)
- James Diebel: **Representing Attitude: Euler Angles, Unit Quaternions, and Rotation Vectors**
- Erik B. Dam, Martin Koch, and Martin Lillholm: **Quaternions, Interpolation and Animation**