Subspace Identification beyond Linear Dynamics

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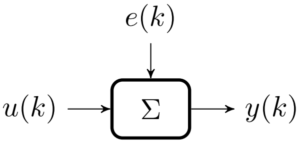
Contents

Problem Definition

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ke(k) \\ y(k) = Cx(k) + e(k) \end{cases}$$



Problem Definition

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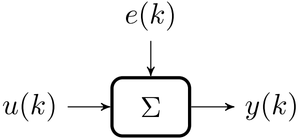


Identification Problem

Given i/o data sequences $\{u(k), y(k)\}_{k=1}^{N}$ from an unknown finite dimensional, linear time-invariant system, determine the system matrices:

$$\Big(A_T, B_T, C_T, D, K_T\Big)$$

and the system order n such that they are a consistent estimate of the orginal system matrices up to a similarity transformation.

The closed-loop SI Strategy (Chiuso, 2007)

3-step strategy

3. If the state sequence $\begin{bmatrix} \hat{x}(t) & \hat{x}(t+1) & \cdots & \hat{x}(t+N-1) \end{bmatrix} = \hat{X}_{t,N}$ is estimated the problem is solved!

The closed-loop SI Strategy (Chiuso, 2007)

3-step strategy

2. Estimate state sequence $\begin{bmatrix} \hat{x}(t) & \hat{x}(t+1) & \cdots & \hat{x}(t+N-1) \end{bmatrix}$ from i/o data and estimated parameters in step 1.

3. If the state sequence $\begin{bmatrix} \hat{x}(t) & \hat{x}(t+1) & \cdots & \hat{x}(t+N-1) \end{bmatrix} = \hat{X}_{t,N}$ is estimated the problem is solved!

The closed-loop SI Strategy (Chiuso, 2007)

3-step strategy

1. Estimate system parameters via linear least-squares.

2. Estimate state sequence $\begin{bmatrix} \hat{x}(t) & \hat{x}(t+1) & \cdots & \hat{x}(t+N-1) \end{bmatrix}$ from i/o data and estimated parameters in step 1.

3. If the state sequence $\begin{bmatrix} \hat{x}(t) & \hat{x}(t+1) & \cdots & \hat{x}(t+N-1) \end{bmatrix} = \hat{X}_{t,N}$ is estimated the problem is solved!

The observer form

The innovation model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{aligned}$$

This model can be written into

The "observer form"

$$\begin{aligned} \hat{x}(k+1) &= (A - KC)\hat{x}(k) + \begin{bmatrix} (B - KD) & K \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \\ y(k) &= C\hat{x}(k) + \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} + e(k) \end{aligned}$$

which is written more compactly as,

The observer form

The innovation model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{aligned}$$

This model can be written into

The "observer form"

$$\begin{aligned} \hat{x}(k+1) &= \Phi\hat{x}(k) + B_z z(k) \\ y(k) &= C\hat{x}(k) + D_z z(k) + e(k) \end{aligned}$$

with $\Phi = (A - KC)$, $B_z = \begin{bmatrix} (B - KD) & K \end{bmatrix}$, $z(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$

and $D_z = \begin{bmatrix} D & 0 \end{bmatrix}$.

Estimating the state sequence

Lemma 1

Let the observer form of the innovation model be given as

$$\begin{cases} \hat{x}(k+1) & = & \Phi \hat{x}(k) + B_z z(k) \\ y(k) & = & C\hat{x}(k) + D_z z(k) + e(k) \end{cases}$$

then

$$\hat{x}(t) = \Phi^p \hat{x}(t-p) + \begin{bmatrix} \Phi^{p-1} B_z & \Phi^{p-2} B_z & \cdots & B_z \end{bmatrix} \underbrace{\begin{bmatrix} z(t-p) \\ z(t-p+1) \\ \vdots \\ z(t-1) \end{bmatrix}}_{\mathcal{Z}_{t-p,p,1}}$$

Estimating the state sequence

Corollary 1

The (A) matrix with the required state sequence as its row space is given by:

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{f-1} \end{bmatrix}}_{\mathcal{O}_f} \hat{X}_{t,N} = \mathcal{O}_f \Phi^p \hat{X}_{t-p,N} +$$

$$\begin{bmatrix} \boxed{C\Phi^{p-1}B_z} & C\Phi^{p-2}B_z & \cdots & CB_z \\ C\Phi^{p}B_z & \boxed{C\Phi^{p-1}B_z} & \cdots & C\Phi B_z \\ \vdots & & \ddots & \vdots \\ C\Phi^{p+f-1}B_z & \cdots & & C\Phi^{f-1}B_z \end{bmatrix} \mathcal{Z}_{t-p,p,N}$$

Estimating the state sequence

Subspace "trick" 1

The matrix Φ is A.S. and therefore: $\boxed{\Phi^\ell \approx 0}$ for $\ell \geq p \quad \Rightarrow$

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{f-1}\end{bmatrix}}_{\mathcal{O}_f} \hat{X}_{t,N} \approx \mathcal{O}_f \Phi^p \hat{X}_{t-p,N} +$$

$$\begin{bmatrix} \boxed{C\Phi^{p-1}B_z} & C\Phi^{p-2}B_z & \cdots & CB_z \\ C\Phi^p B_z & \boxed{C\Phi^{p-1}B_z} & \cdots & C\Phi B_z \\ \vdots & & \ddots & \vdots \\ C\Phi^{p+f-1}B_z & \cdots & & C\Phi^{f-1}B_z \end{bmatrix} \mathcal{Z}_{t-p,p,N}$$

Estimating the state sequence

Subspace "trick" 2: From Lemma 1

Let the observer form of the innovation model be given as

$$\begin{cases} \hat{x}(k+1) & = & \Phi\hat{x}(k) + B_z z(k) \\ y(k) & = & C\hat{x}(k) + D_z z(k) + e(k) \end{cases}$$

then

$$y(t) = C\Phi^p\hat{x}(t-p) + \begin{bmatrix} C\Phi^{p-1}B_z & C\Phi^{p-2}B_z & \cdots & CB_z \end{bmatrix} \mathcal{Z}_{t-p,p,1} + Du(t)$$

With "trick" 1:

$$y(t) \approx C\Phi^p\hat{x}(t-p) + \begin{bmatrix} C\Phi^{p-1}B_z & C\Phi^{p-2}B_z & \cdots & CB_z \end{bmatrix} \mathcal{Z}_{t-p,p,1} + Du(t)$$

The system parameters $C\Phi^j B_z$ for $j = 0, \cdots, p-1$ can be (accurately) approximated via solving a linear least squares problem (step 1!)

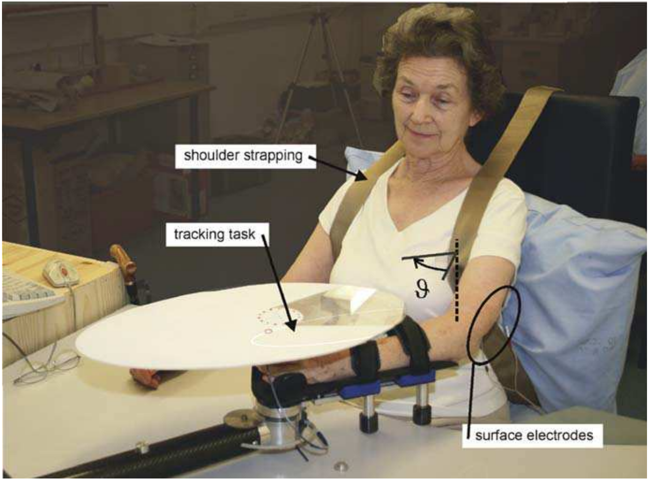
Subspace "trick" 1 and 2 combined

The matrix Φ is A.S. and therefore: $\boxed{\Phi^\ell \approx 0}$ for $\ell \geq p \quad \Rightarrow$

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{f-1} \end{bmatrix}}_{\mathcal{O}_f} \hat{X}_{t,N} \approx \mathcal{O}_f \Phi^p \hat{X}_{t-p,N} +$$

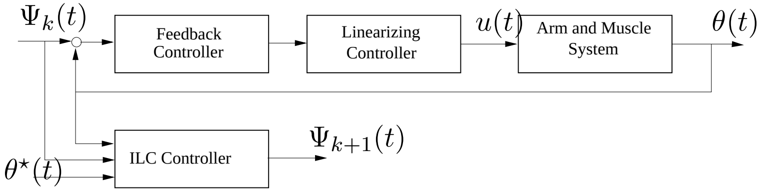
$$\begin{bmatrix} C\Phi^{p-1}B_z & C\Phi^{p-2}B_z & \cdots & CB_z \\ C\Phi^{p}B_z & C\Phi^{p-1}B_z & \cdots & C\Phi B_z \\ \vdots & & \ddots & \vdots \\ C\Phi^{p+f-1}B_z & \cdots & & C\Phi^{f-1}B_z \end{bmatrix} \mathcal{Z}_{t-p,p,N}$$

A stroke patient at a robotic workstation



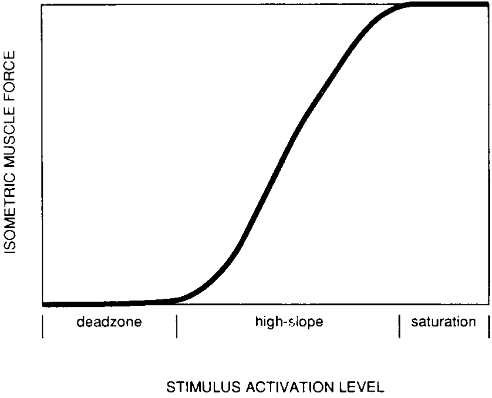
(Courtesy: Fengmin Le, et. al. (Univeristy of Southhampton, 2010))

Control objective for the robotic workstation



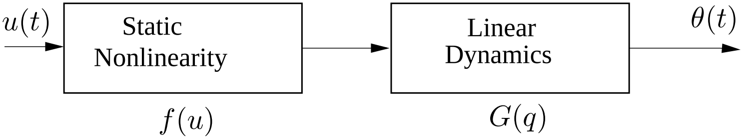
(Courtesy: Fengmin Le, et. al. (Univeristy of Southhampton, 2010))

The Isometric Recruitment Curve



(From: Duree, MacLean, 1989))

Arm Muscle System



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A closed-loop Hammerstein SI identifcation problem

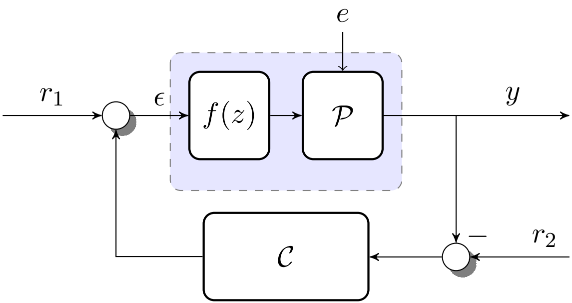

Figure: The closed-loop Hammerstein configuration

Multivariable Spline Approximation of $f(z)$

Step 1: Triangulation

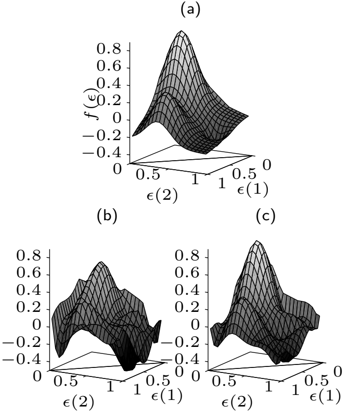

Figure: Franke's test function $f(\epsilon)$ (a)

The nonlinear mapping $f(\epsilon)$ is approximated as:

$$f(\epsilon) \approx \mathcal{C}^{\top}\mathcal{B}(\epsilon_k) \in \mathbb{R}^m$$

(See G. van der Veen, J.W. van Wingerden and M. Verhaegen, 2012; for more details).

Exploiting model structure?

With the spline approximation the Hammerstein model (to be identified) can be written as:

$$\begin{cases} x_{k+1} &= Ax_k + B_{\mathsf{l}} u_k + B_{\mathsf{nl}} f(\epsilon_k) + K e_k, \\ y_k &= Cx_k + e_k, \end{cases} \overset{\approx}{\Longrightarrow}$$

$$\begin{cases} x_{k+1} &= Ax_k + B_{\mathsf{l}} u_k + B_{\mathsf{nl}} \mathcal{C}^\top \underbrace{\mathcal{B}(\epsilon_k)}_{known\ for\ given\ \epsilon_k} + K e_k, \\ y_k &= Cx_k + e_k, \end{cases}$$

Step 2: Hammerstein Model Identification

Exploiting model structure?

With the spline approximation the Hammerstein model (to be identified) can be written as:

$$\begin{cases} x_{k+1} & = & Ax_k + B_{\mathsf{l}} u_k + B_{\mathsf{nl}} f(\epsilon_k) + K e_k, \\ y_k & = & C x_k + e_k, \end{cases} \overset{\approx}{\Longrightarrow}$$

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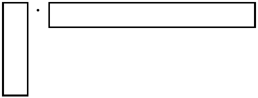
The key observation is that the matrix

$$B_{\mathsf{nl}} \mathcal{C}^\top = $$



has low rank!

Step 2: Structured Hammerstein Model Identification

Given i/o data sequences $\{u_k, z_k, y_k\}_{k=1}^{N}$ from an unknown finite dimensional, Hammerstein, time-invariant system, parametrized as:

$$\begin{cases} x_{k+1} = Ax_k + B_{\mathsf{l}} u_k + \underbrace{B_{\mathsf{nl}} \mathcal{C}^\top}_{\breve{B}_{\mathsf{nl}}} \mathcal{B}(\epsilon_k) + Ke_k, \\ y_k = Cx_k + e_k, \end{cases}$$

determine the system matrices:

$$\left(A_T, B_{\mathsf{l},T}, C_T, D, K_T, \breve{B}_{\mathsf{nl},T} \right)$$

and the system order n such that they are a consistent estimate of the orginal system matrices up to a similarity transformation. Problem: The matrix $\breve{B}_{\mathsf{nl},T}$ contains "lots of redundant elements"!

Step 2: Structured Hammerstein Model Identification

Given i/o data sequences $\{u_k, z_k, y_k\}_{k=1}^{N}$ from an unknown finite dimensional, Hammerstein, time-invariant system, parametrized as:

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determine the system matrices:

$$\left(A_T, B_{\mathsf{l},T}, C_T, D, K_T, \breve{B}_{\mathsf{nl},T} \right)$$

and the system order n such that they are a consistent estimate of the orginal system matrices up to a similarity transformation. Solution: Additional constraint that $\breve{B}_{\mathsf{nl},T}$ "has rank m"!

Step 2: rank constraint

Nuclear Norm

Definition: The nuclear norm of a matrix A is given as:

$$\|A\|_* = \sum_{i=1}^{\min(m,n)} \sigma_i(A).$$

When added this constraint to a linear least squares problem, a convex optimization problem results.

Results $N = 2000$.

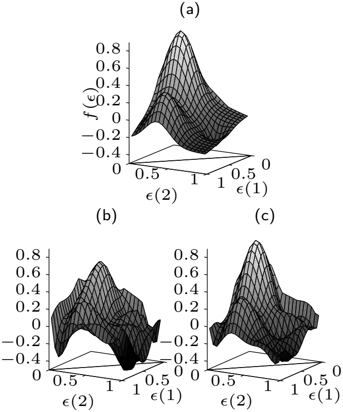


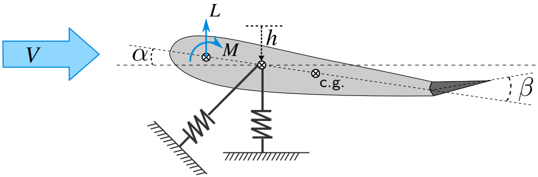
Figure: Franke's test function $f(\epsilon)$

$$
\begin{aligned}
y_k &= G(\check{z})u_k + H(\check{z})e_k, \\
u_k &= f(\epsilon_k), \\
G(\check{z}) &= \frac{0.2571\check{z}^3 - 0.2034\check{z}^2 - 0.19}{\check{z}^4 - 3.421\check{z}^3 + 4.838vz^2 - 3} \\
H(\check{z}) &= \frac{0.3454\check{z} + 0.2846}{\check{z}^2 - 0.9381\check{z} + 0.5681}.
\end{aligned}
$$

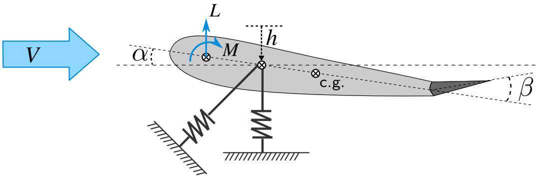
with,

$$
\begin{aligned}
e_k &\sim \mathcal{N}(0, 0.5), \\
\epsilon_k &\sim \mathcal{U}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)
\end{aligned}
$$

System overview: smart 2-D airfoil



System overview: smart 2-D airfoil



- Equations of motion:

$$\mathrm{M}\begin{bmatrix}\ddot{h}\\ \ddot{\alpha}\end{bmatrix}+\mathrm{C}\begin{bmatrix}\dot{h}\\ \dot{\alpha}\end{bmatrix}+\mathrm{K}\begin{bmatrix}h\\ \alpha\end{bmatrix}=\begin{bmatrix}-L\\ M\end{bmatrix}$$

- Nonlinear aerodynamics:

$$L = q_2\begin{bmatrix}0, & V^2, & V, & q_6V,\end{bmatrix}\begin{bmatrix}h\\ \alpha\\ \dot{h}\\ \dot{\alpha}\end{bmatrix}+q_5V^2\beta,$$ and similarly for M.

LPV modeling

- Analytical model: continuous-time affine LPV

$$\begin{cases} \dot{x} = \left(A_1 + A_2 V + A_3 V^2\right) x + \left(B_3 V^2\right) \beta \\ y = C_1 x \end{cases}$$

with $x = \begin{bmatrix} h, & \alpha, & \dot{h}, & \dot{\alpha} \end{bmatrix}^T$, $y = \begin{bmatrix} h, & \alpha \end{bmatrix}^T$.

LPV modeling

- Analytical model: continuous-time affine LPV

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with $x = \begin{bmatrix} h, & \alpha, & \dot{h}, & \dot{\alpha} \end{bmatrix}^T$, $y = \begin{bmatrix} h, & \alpha \end{bmatrix}^T$.

- With discretization, we lose the affine structure:

$$A_d\left(V, V^2, \frac{1}{V}, \cdots\right) = \left(I + \frac{T_s}{2} A(V, V^2)\right)\left(I - \frac{T_s}{2} A(V, V^2)\right)^{-1},$$

$$B_d\left(V, V^2, \frac{1}{V}, \cdots\right) = \sqrt{T_s}\left(I - \frac{T_s}{2} A(V, V^2)\right)^{-1} B(V^2),$$

and similarly for C_d, D_d(!)

LPV modeling

- Analytical model: continuous-time affine LPV

$$\begin{cases} \dot{x} = \left(A_1 + A_2 V + A_3 V^2\right) x + \left(B_3 V^2\right) \beta \\ y = C_1 x \end{cases}$$

with $x = \begin{bmatrix} h, & \alpha, & \dot{h}, & \dot{\alpha} \end{bmatrix}^T$, $y = \begin{bmatrix} h, & \alpha \end{bmatrix}^T$.

- With discretization, we lose the affine structure:

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and similarly for C_d, D_d(!)

- In identification, we 'force' affine dependence, with:

$$\mu_k = \begin{bmatrix} 1, & V_k, & V_k^2 \end{bmatrix}^T$$

LPV model structures

LPV system (innovation form):

Affine parameter dependence:

$$x_{k+1} = \sum_{i=1}^{m} \mu_k^{(i)} \left(A^{(i)} x_k + B^{(i)} u_k + K^{(i)} e_k \right),$$
$$y_k = C x_k + D u_k + e_k,$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^r$, $y_k \in \mathbb{R}^\ell$, $\mu_k \in \mathbb{R}^m$ and $\mu_k^{(1)} = 1$.

LPV model structures

LPV system (innovation form):

Affine parameter dependence:

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{m} \mu_k^{(i)} \left(A^{(i)} x_k + B^{(i)} u_k + K^{(i)} e_k \right), \\ y_k &= C x_k + D u_k + e_k, \end{aligned}$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^r$, $y_k \in \mathbb{R}^\ell$, $\mu_k \in \mathbb{R}^m$ and $\mu_k^{(1)} = 1$.

LPV system (predictor form):

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{m} \mu_k^{(i)} \left(\tilde{A}^{(i)} x_k + \tilde{B}^{(i)} u_k + K^{(i)} y_k \right) \\ y_k &= C x_k + D u_k + e_k, \end{aligned}$$

with $\tilde{A}^{(i)} = A^{(i)} - K^{(i)} C, \quad \tilde{B}^{(i)} = B^{(i)} - K^{(i)} D.$

Problem formulation

LPV system (predictor form):

$$
\begin{aligned}
x_{k+1} &= \sum_{i=1}^{m} \mu_k^{(i)} \left(\tilde{A}^{(i)} x_k + \tilde{B}^{(i)} u_k + K^{(i)} y_k \right) \\
y_k &= C x_k + D u_k + e_k,
\end{aligned}
$$

Problem formulation

LPV system (predictor form):

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{m} \mu_k^{(i)} \left(\tilde{A}^{(i)} x_k + \tilde{B}^{(i)} u_k + K^{(i)} y_k \right) \\ y_k &= C x_k + D u_k + e_k, \end{aligned}$$

Problem statement:

Given u_k, y_k, μ_k for $k = \{1, \ldots, N\}$, find system matrices $\left\{ \tilde{A}^{(i)}, \tilde{B}^{(i)}, \tilde{K}^{(i)} \right\}_{i=1}^{m}$,$C$,$D$ up to a similarity transformation.

Intermediate step: find the state sequence.

Estimating the state sequence

Lemma 1

For the given LPV system and the definition of the joint input-output data vector $z(t) = \left[u(t)^T \quad y(t)^T \right]^T$, and the selection of a past window $p >$ system order:

$$x_{k+p} = \underbrace{\tilde{A}\left(\mu_{k+p-1}\right)\tilde{A}\left(\mu_{k+p-2}\right)\cdots\tilde{A}\left(\mu_{k}\right)}_{=\phi_{p,k}\approx 0 \text{ for large } p} x_k + \mathcal{K}^p N_k^p \bar{z}_k^p$$

with Regressor $N_k^p \bar{z}_k^p$:

$$N_k^p \bar{z}_k^p = \begin{bmatrix} \mu_{k+p-1} \otimes \ldots \otimes \mu_{k+1} \otimes \mu_k \otimes \begin{bmatrix} u_k \\ y_k \end{bmatrix} \\ \mu_{k+p-1} \otimes \ldots \otimes \mu_{k+1} \otimes \begin{bmatrix} u_{k+1} \\ y_{k+1} \end{bmatrix} \\ \vdots \\ \mu_{k+p-1} \otimes \begin{bmatrix} u_{k+p-1} \\ y_{k+p-1} \end{bmatrix} \end{bmatrix}$$

The 3-step procedure

1. Solve linear problem: $\min_{C\mathcal{K}^p, D} ||Y - C\mathcal{K}^p Z - DU||_F^2$, with $Y = [y_{p+1}, \cdots, y_N], U = [u_{p+1}, \cdots, u_N], Z = \left[N_1^p \bar{z}_1^p, \cdots, N_{N-p}^p \bar{z}_{N-p}^p\right]$
2. ...
3. ...

The 3-step procedure

1. Solve linear problem: $\min_{C\mathcal{K}^p, D} ||Y - C\mathcal{K}^p Z - DU||_F^2$, with
$Y = [y_{p+1}, \cdots, y_N], U = [u_{p+1}, \cdots, u_N], Z = \left[N_1^p \bar{z}_1^p, \cdots, N_{N-p}^p \bar{z}_{N-p}^p\right]$

2. Construct:
$$\mathcal{O}^p\mathcal{K}^p = \begin{bmatrix} C\mathcal{L}_p & C\mathcal{L}_{p-1} & \cdots & C\mathcal{L}_1 \\ C\tilde{A}^{(1)}\mathcal{L}_p & C\tilde{A}^{(1)}\mathcal{L}_{p-1} & \cdots & C\tilde{A}^{(1)}\mathcal{L}_1 \\ \vdots & \vdots & \ddots & \vdots \\ C\left(\tilde{A}^{(1)}\right)^{p-1}\mathcal{L}_p & C\left(\tilde{A}^{(1)}\right)^{p-1}\mathcal{L}_{p-1} & \cdots & C\left(\tilde{A}^{(1)}\right)^{p-1}\mathcal{L}_1 \end{bmatrix}$$

3. …

The 3-step procedure

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2. Construct:
$$\mathcal{O}^p\mathcal{K}^p \approx \begin{bmatrix} C\mathcal{L}_p & C\mathcal{L}_{p-1} & \cdots & C\mathcal{L}_1 \\ 0 & C\tilde{A}^{(1)}\mathcal{L}_{p-1} & \cdots & C\tilde{A}^{(1)}\mathcal{L}_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C\left(\tilde{A}^{(1)}\right)^{p-1}\mathcal{L}_1 \end{bmatrix}$$
based on $\phi_{p,k} \approx 0$
3. ...

The 3-step procedure

1. Solve linear problem: $\min_{C\mathcal{K}^p, D} ||Y - C\mathcal{K}^p Z - DU||_F^2$, with
$Y = [y_{p+1}, \cdots, y_N], U = [u_{p+1}, \cdots, u_N], Z = \left[N_1^p \bar{z}_1^p, \cdots, N_{N-p}^p \bar{z}_{N-p}^p\right]$

2. Construct:

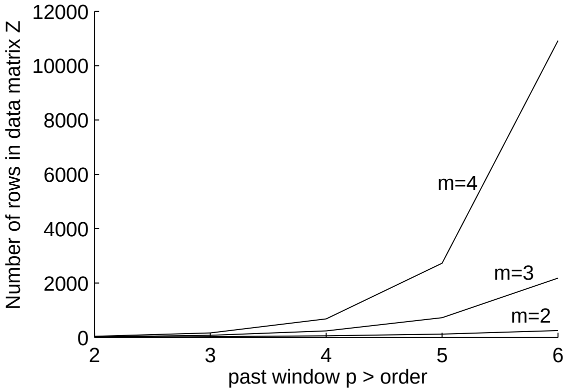
$$\mathcal{O}^p \mathcal{K}^p \approx \begin{bmatrix} C\mathcal{L}_p & C\mathcal{L}_{p-1} & \cdots & C\mathcal{L}_1 \\ 0 & C\tilde{A}^{(1)}\mathcal{L}_{p-1} & \cdots & C\tilde{A}^{(1)}\mathcal{L}_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C\left(\tilde{A}^{(1)}\right)^{p-1}\mathcal{L}_1 \end{bmatrix}$$

based on $\phi_{p,k} \approx 0$

3. SVD decompose to find state sequence:

$$\widehat{\mathcal{O}^p \mathcal{K}^p} Z = \mathcal{O}^p X$$

Curse of dimensionality



Kernel method (van Wingerden, Verhaegen)

1. Construct data matrices $Y, Z^T Z \in \mathbb{R}^{N \times N}$ from $\{u_k, y_k, \mu_k\}_{k=1}^N$.
2. Parameter estimation problem:

$$\min_{\alpha} \left(\left\| Y - \alpha Z^T Z \right\|_2^2 \right)$$

3. $\alpha \overset{construct}{\rightarrow} \Gamma_p \mathcal{K}_p Z \overset{SVD}{\rightarrow}$ state sequence $\overset{LLS}{\rightarrow} \{A_i, B_i\}_{i=1}^M, C$

- ⊕ Numerically more efficient.
- ⊖ Variance error on estimated parameters, **regularization needed**.

Kernel method with nuclear norm regularization

1. Construct data matrices $Y, Z^T Z \in \mathbb{R}^{N \times N}$ from $\{u_k, y_k, \mu_k\}_{k=1}^N$.
2. Parameter estimation problem with NN[2] regularization:

$$\min_{\alpha} \left(\left\| Y - \alpha Z^T Z \right\|_F^2 + \lambda \| \Gamma_p \mathcal{K}_p Z \|_* \right)$$

$$\|\cdot\|_* = \text{sum of singular values}$$

3. $\alpha \overset{construct}{\rightarrow} \Gamma_p \mathcal{K}_p Z \overset{SVD}{\rightarrow}$ state sequence $\overset{LLS}{\rightarrow} \{A_i, B_i\}_{i=1}^M, C$

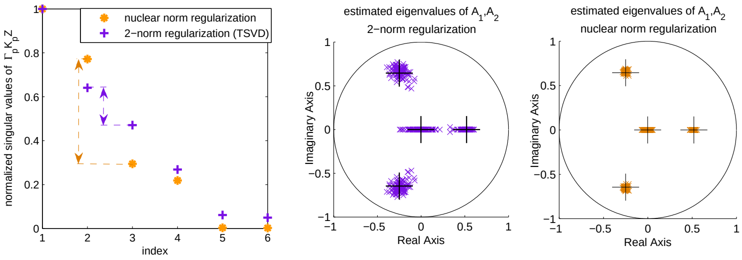
[2] Nuclear Norm

Results

Simulation example low order system:

2nd order system, 1 input, 2 outputs, 1 scheduling variable.

80 samples, SNR=75dB, kernel method



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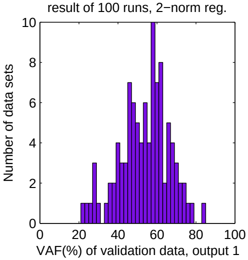
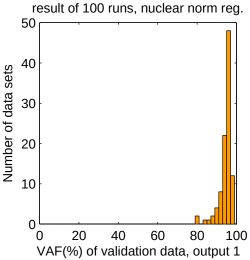
$$VAF = \left(1 - \frac{\mathrm{var}(\hat{y}_k - y_k)}{\mathrm{var}(y_k)}\right) \cdot 100\%$$





Conclusions for now

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2. But ... The computational challenges are great!
3. The engineering rewards are idem dito: Accurate model info with convex optimization!