Filtering and Identification

Day 5 - Lecture 1:
Subspace Identification
Instrumental Variables

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Overview

- **Recap: Subspace Identification with additive white noise disturbances**
- Identification of Output Error Models
- Identification of Innovation Models
- Closed-loop identification
- Estimating the Kalman Gain
- Identification and Control: Experimental Results.

The Deterministic Case

LTI System: $\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$

Data equation $Y_{0,s,N} = \mathcal{O}_s X_{0,N} + \mathcal{T}_s U_{0,s,N}$

Recovery column space of extended observability matrix $\mathcal{O}_s$:

Let, $\Pi^{\perp}_{U_{0,s,N}} = \left(I - U^T_{0,s,N} \left(U_{0,s,N} U^T_{0,s,N} \right)^{-1} U_{0,s,N} \right)$, then:

$$Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}} = \mathcal{O}_s X_{0,N}\Pi^{\perp}_{U_{0,s,N}} \Rightarrow \text{range}(Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}) \stackrel{?}{=} \text{range}(\mathcal{O}_s)$$

The RQ factorization provides an efficient solution $\Rightarrow$:

$$\begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}}_{Q} \quad QQ^T = I \quad \boxed{\Rightarrow Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}} = R_{22}Q_2}$$

The 5-line matlab solution:Basic MOESP

GIVEN: The i/o data sequences $\{u(k), y(k)\}_{k=0}^{N-1}$ and the integer s to specify number of block rows of the Hankel matrices $U_{0,s,N}, Y_{0,s,N}$ ($s > n$?).

THEN DO:

- Construct Hankel matrices $U_{0,s,N}, Y_{0,s,N}$ `(U,Y)`
- RQ factorization: `r = triu(qr([U ; Y]'))';`
- Extract `R22`
- Range (column space) calculation + order detection:

```
[uu,ss,vv]=svd(R22);
semilogy(diag(ss),'xr');
```

- Estimate A_T, C_T

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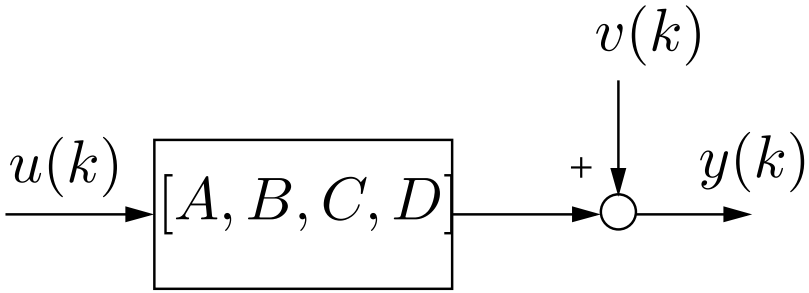
The Output-error identification problem

Additive Coloured Noise



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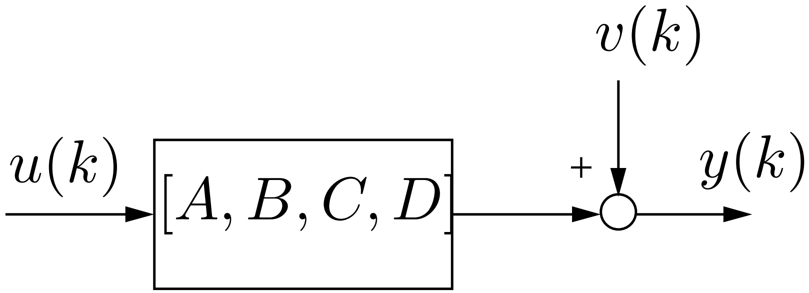

Problem:

$$\{u(k), y(k)\}_{k=0}^{N-1} \qquad \Longrightarrow$$

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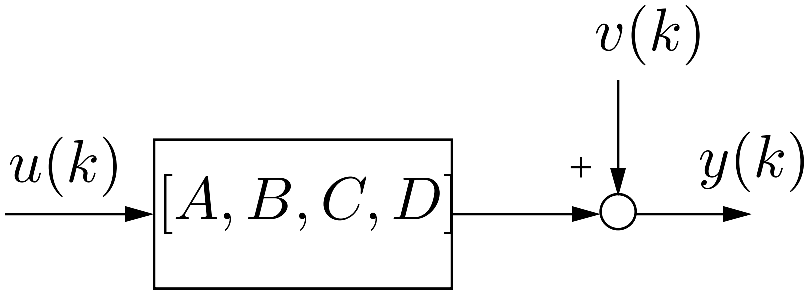

Problem:

$$\{u(k), y(k)\}_{k=0}^{N-1} \Longrightarrow$$

$$\Big([A, B, C, D]\,, x(0) \Big)_T$$ consistently

Assumptions about the additive disturbance $v(k)$:

- zero-mean
- WSS stochastic process — Rational Spectrum
- uncorrelated with the input $u(k)$

tst_sub.m

Consistent Estimation via Instrumental Variables

Problem: When $v(k)$ is coloured noise, i.e. $\frac{1}{N}V_{0,s,N}V_{0,s,N}^T \neq \sigma_v^2 I$, it can be shown that:

$$\text{range}(\lim_{N\to\infty} \frac{1}{\sqrt{N}} Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}) \not\subset \text{range}(\mathcal{O}_s)$$

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Solution: Remove $V_{0,s,N}$ from the data equation.

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Projected data eq. $\quad Y_{0,s,N} \Pi^{\perp}_{U_{0,s,N}} = \mathcal{O}_s X_{0,N} \Pi^{\perp}_{U_{0,s,N}} + V_{0,s,N} \Pi^{\perp}_{U_{0,s,N}}$

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Instrumental Variables: $Y_{0,s,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T = \mathcal{O}_s \underbrace{X_{0,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T}_{\boxed{1}} + \underbrace{V_{0,s,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T}_{\boxed{2}}$

Two requirements on the instrumental variable matrix Z_N!

$\boxed{1}$ rank $\left(\lim_{N\to\infty} \frac{1}{N} X_{0,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T\right) = n$ $\quad \boxed{2}$ $\lim_{N\to\infty} \frac{1}{N} V_{0,s,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T = 0$

A first choice for Z_N?

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Since $u(k)$ and $v(k)$ are independent, a possible choice for Z_N equals:

$$Z_N = U_{0,s,N}$$

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Two requirements on the instrumental variable matrix Z_N!

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A first choice for Z_N?

Two requirements on the instrumental variable matrix Z_N!

[1] rank $\left(\lim_{N\to\infty} \frac{1}{N} X_{0,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T\right) = n$ [2] $\lim_{N\to\infty} \frac{1}{N} V_{0,s,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T = 0$

Since $u(k)$ and $v(k)$ are independent, a possible choice for Z_N equals:

$$Z_N = U_{0,s,N}$$

- [2] $\lim_{N\to\infty} \frac{1}{N} V_{0,s,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T = 0$
- [1] rank $\left(\lim_{N\to\infty} \frac{1}{N} X_{0,N} \Pi^{\perp}_{U_{0,s,N}} Z_N^T\right) = 0$

Splitting up the data

For the output sequence $\{y(k)\}_{k=0}^{N+2s-2}$ we can define the Hankel matrices (past and future):

$$Y_{\boxed{0},s,N} = \begin{bmatrix} y(\boxed{0}) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \\ y(s-1) & y(s) & \cdots & y(N+s-2) \end{bmatrix} \quad \text{past}$$

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.

$$Y_{\boxed{s},s,N} = \begin{bmatrix} y(\boxed{s}) & y(s+1) & \cdots & y(N+s-1) \\ y(s+1) & y(s+2) & & y(N+s) \\ \vdots & & \ddots & \\ y(2s-1) & y(s) & \cdots & y(N+2s-2) \end{bmatrix} \quad \text{future}$$

idem for the input $u(k)$, etc

Instrumental variables for Output-Error Problem

Consider the Projected data eq for the “future” matrices.

$$Y_{s,s,N}\Pi^{\perp}_{U_{s,s,N}} = \mathcal{O}_s X_{s,N}\Pi^{\perp}_{U_{s,s,N}} + V_{s,s,N}\Pi^{\perp}_{U_{s,s,N}}$$

and take Z_N equal to the “past” $U_{0,s,N}$, then we have:

$$\lim_{N\to\infty} \frac{1}{N} V_{s,s,N}\Pi^{\perp}_{U_{s,s,N}} U^T_{0,s,N} = 0$$

and we hope that the following condition is satisfied:

Instrumental variables for Output-Error Problem

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and we hope that the following condition is satisfied:

$$\text{rank}\left(\lim_{N\to\infty}\frac{1}{N}X_{s,N}\Pi^{\perp}_{U_{s,s,N}}U^T_{0,s,N}\right) = n$$

$\Rightarrow$ **PI-MOESP method**

RQ for efficient implementation

Let the following RQ data compression be given:

$$\begin{bmatrix} U_{\boxed{s},s,N} \\ U_{\boxed{0},s,N} \\ Y_{\boxed{s},s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

Lemma: Given the RQ factorization, we have:

1. $\left(Y_{\boxed{s},s,N} \Pi^{\perp}_{U_{\boxed{s},s,N}} U_{\boxed{0},s,N}^{T} \right) = R_{32} R_{22}^T.$
2. As a result:

$$\text{range}\left(\lim_{N \to \infty} \frac{1}{N} R_{32} R_{22}^T \right) \subset \text{range}(\mathcal{O}_s)$$

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Can we do better?

Can we do better?

Yes, we can

Can we do better?

Yes, we can

When more information is available about the noise!

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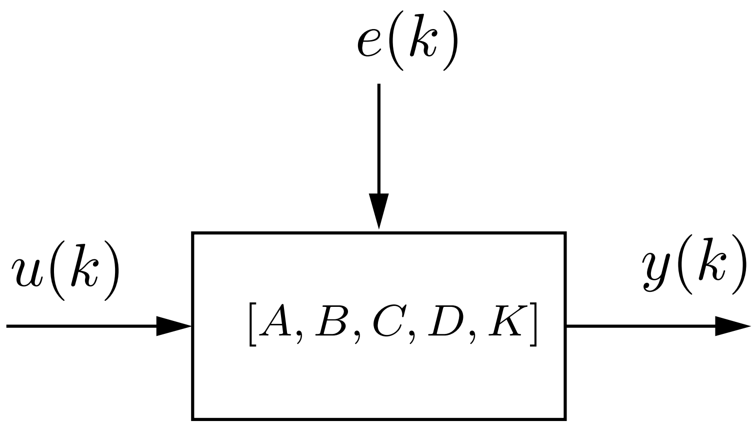
The Innovation Model identification problem

The Innovation Model



$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{cases}$$

Assumptions about the innovation $e(k)$:

- zero-mean
- white-noise stochastic process
- uncorrelated with the input $u(k)$

Be-aware that the innovation model often is an “augmented” model

The Data Equation for the Innovation Model

With the innovation model given as:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{cases}$$

then the data equations reads:

$$Y_{i,s,N} = \mathcal{O}_s X_{i,N} + \mathcal{T}_{u,s} U_{i,s,N} + \mathcal{T}_{e,s} E_{i,s,N}$$

Condition 4: $x(k)$ and $e(k+\ell)$ are uncorrelated for $\ell > 0$

Statistically: **Lemma:** Let the innovation model be given:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{cases}$$

with $e(k+\ell)$ a zero-mean white noise sequence, indepedent of $u(k), x(0)\ \forall k, \ell$, then,

$$\begin{aligned} E[x(k)e^T(k+\ell)] &= E[e(k+\ell)x^T(k)] = 0 \quad \ell \geq 0 \\ E[y(k)e^T(k+\ell)] &= E[e(k+\ell)y^T(k)] = 0 \quad \ell > 0 \end{aligned}$$

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Proof: $x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^j Bu(k-j-1) + \sum_{j=0}^{k-1} A^j Ke(k-j-1)$. Therefor,

$$\begin{aligned} E[x(k)e^T(k+\ell)] &= A^k E[x(0)e^T(k+\ell)] + \sum_{j=0}^{k-1} A^j \Big(BE[u(k-j-1)e^T(k+\ell)] + \cdots \\ &\quad \cdots KE[e(k-j-1)e^T(k+\ell)]\Big) \\ &= 0 \end{aligned}$$

Condition 4: $x(k)$ and $e(k+\ell)$ are uncorrelated for $\ell > 0$ (C'td)

Algebraically (by ergodicity): Consider the data equations for $i = s$:

$$Y_{s,s,N} = \mathcal{O}_s X_{s,N} + \mathcal{T}_{u,s} U_{s,s,N} + \mathcal{T}_{e,s} E_{,ss,N}$$

then,

$$\Leftrightarrow \lim_{N\to\infty} \frac{1}{N} \begin{bmatrix} e(s) & e(s+1) & \cdots & e(s+N-1) \\ e(s+1) & & & \\ & & \ddots & \\ e(2s-1) & & \vdots & \end{bmatrix} \begin{bmatrix} y^T(0) & y^T(1) & \cdots & y^T(s-1) \\ y^T(1) & & \ddots & \\ \vdots & & & \\ y^T(N-1) & & & \end{bmatrix}$$

$$\Leftrightarrow \lim_{N\to\infty} \frac{1}{N} E_{s,s,N} Y_{0,s,N} = 0$$

Instrumental variables for Output-Error Problem

Consider the Projected data eq.

$$Y_{s,s,N}\Pi^{\perp}_{U_{s,s,N}} = \mathcal{O}_s X_{s,N}\Pi^{\perp}_{U_{s,s,N}} + E_{s,s,N}\Pi^{\perp}_{U_{s,s,N}}$$

and take Z_N equal to $\begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}$, then we have the “PO consistency condition”:

$$\lim_{N\to\infty} \frac{1}{N} E_{s,s,N}\Pi^{\perp}_{U_{s,s,N}} \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}^T = 0$$

and we hope that the following condition is satisfied:

$$\text{rank}\left(\lim_{N\to\infty} \frac{1}{N} X_{s,N}\Pi^{\perp}_{U_{s,s,N}} \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}^T\right) = n$$

$\Rightarrow$ **PO-MOESP method**

RQ for efficient implementation:

$$\begin{bmatrix} U_{\boxed{s},s,N} \\ \begin{bmatrix} U_{\boxed{0},s,N} \\ Y_{\boxed{0},s,N} \end{bmatrix} \\ Y_{\boxed{s},s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

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2. As a result:

$$\text{range}\left(\lim_{N\to\infty} \frac{1}{N} R_{32} R_{22}^T \right) \subset \text{range}(\mathcal{O}_s)$$

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Consider the innovation model:

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1.5 & -0.7 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} e(k) \\
y(k) &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(k) + e(k)
\end{aligned}
$$

In the experiment we can vary the number of data samples N, the number of block rows s of the Hankel matrices. The total number of trials is 100.

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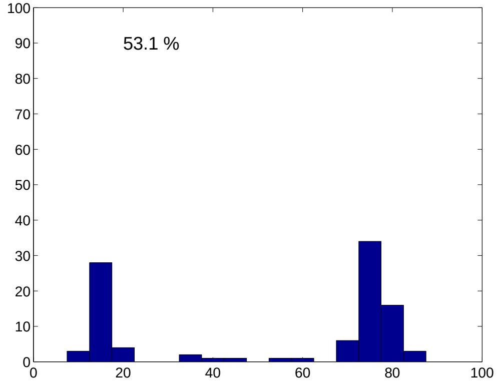
Consider the innovation model:

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1.5 & -0.7 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1.69 & -0.96 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e(k) \\
y(k) &= \begin{bmatrix} 1 & -1 & 0.69 & -0.76 \end{bmatrix} x(k) + e(k)
\end{aligned}
$$

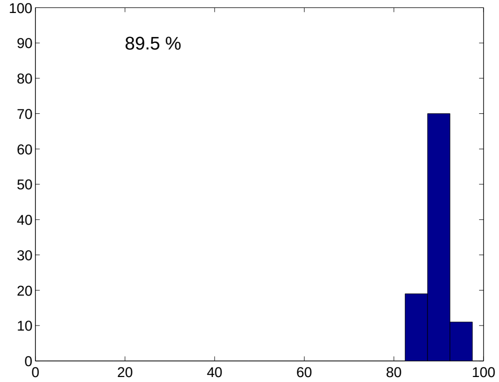
In the experiment we can vary the number of data samples N, the number of block rows s of the Hankel matrices and the model order! The total number of trials is 100.

A happy mariage

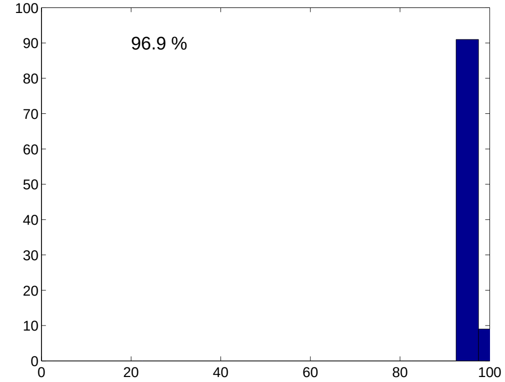
Distribution of the VAF values
for 100 identification experiments
on the acoustical duct



PEM



SI



SI and PEM

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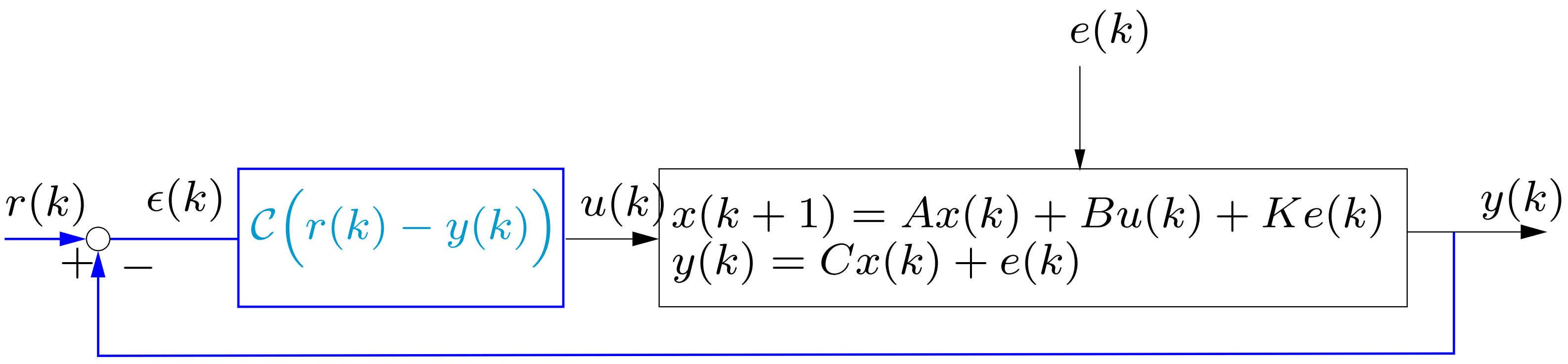
Closed-loop Configuration



Then under the assumptions of causal plant and controller:

- $E[u(k)e(k+\ell)^T] = 0$ for $\ell \geq 0$ if there is *at least one delay in the loop*.
- $E[x(k)e(k+\ell)^T] = 0$ for $\ell \geq 0$.
- $E[y(k)e(k+\ell)^T] = 0$ for $\ell > 0$.

Consistency in closed-loop?

Recall the open-loop “PO consistency condition”,

$$\lim_{N\to\infty} \frac{1}{N} E_{s,s,N} \Pi^{\perp}_{U_{s,s,N}} \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}^T = 0?$$

When we have that,

- $E[u(k)e(k+\ell)^T] = 0$ for $\ell \geq 0$ if there is *at least one delay in the loop*.
- $E[y(k)e(k+\ell)^T] = 0$ for $\ell > 0$.

Why is the “PO consistency condition” though not valid in *closed-loop*?

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Estimating the Kalman Gain from i-o data

Given the innovation model:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + e(k) \qquad \text{with } K \textbf{ Kalman gain} \\ \Rightarrow \hat{x}(k+1) &= \underbrace{(A-KC)}_{\Phi}\hat{x}(k) + Bu(k) + Ky(k) \end{aligned}$$

From this it follows that,

$$\hat{x}(j+s) = \Phi^s \hat{x}(j+0) + \underbrace{\begin{bmatrix} \Phi^{s-1}B & \cdots & B & | & \Phi^{s-1}K & \cdots & K \end{bmatrix}}_{\mathcal{L}^s} \begin{bmatrix} u(j+0) \\ \vdots \\ u(j+s-1) \\ \hline y(j+0) \\ \vdots \\ y(j+s-1) \end{bmatrix}$$

for $s = 1, 2, \cdots$ and $j = k, \cdots$.

How to get info on $\mathcal{L}^s$ and therefore on the Kalman gain K?

Estimating the Kalman Gain from i-o data

$$\hat{x}(s+j) \approx \mathcal{L}^s \begin{bmatrix} u(0+j) \\ \vdots \\ u(s-1+j) \\ \hline y(0+j) \\ \vdots \\ y(s-1+j) \end{bmatrix} \Rightarrow \hat{X}_{s,N} = [\hat{x}(s) \quad \hat{x}(s+1) \quad \cdots] \approx \mathcal{L}^s \begin{bmatrix} u(0) & u(1) & \cdots \\ \vdots & & \\ u(s-1) & u(s) & \cdots \\ \hline y(0) & y(1) & \cdots \\ \vdots & & \\ y(s-1) & y(s) & \end{bmatrix} = \mathcal{L}^s \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}$$

Consider the data equation:

$$Y_{s,s,N} = \mathcal{O}_s \hat{X}_{s,N} + \mathcal{T}_s U_{s,s,N} + V_{s,s,N} \approx \underbrace{\mathcal{O}_s \mathcal{L}^s}_{L^z} \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} + \underbrace{\mathcal{T}_s}_{L^u} U_{s,s,N} + V_{s,s,N}$$

Estimating the Kalman Gain from i-o dat

Consider the following linear least squares problem:

$$\begin{bmatrix}\widehat{L}_N^u & \widehat{L}_N^z\end{bmatrix} = \text{arg} \quad \text{min}_{L^u,L^z} \left\| Y_{s,s,N} - \begin{bmatrix}L^u & L^z\end{bmatrix} \begin{bmatrix} U_{s,s,N} \\ \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} \end{bmatrix} \right\|$$

then,

$$\lim_{N\to\infty} \widehat{L}_N^z = \mathcal{O}_s \mathcal{L}^s + \mathcal{O}_s \underbrace{((A - KC))}_{\Phi}{}^s \Delta$$

with

$$\mathcal{L}^s = \begin{bmatrix} \Phi^{s-1}B & \Phi^{s-2}B & \cdots & B & | & \Phi^{s-1}K & \Phi^{s-2}K & \cdots & K \end{bmatrix}$$

Estimating all system matrices

Knowing $\widehat{L}_N^z = U_n \Sigma_n V_n^T + \text{bias}$, we have an estimate of $\widehat{\mathcal{L}}^s = V_n^T$ and hence of the Kalman filter state sequence:

$$\hat{X}_{s,N} = [\hat{x}(s) \quad \hat{x}(s+1) \quad \cdots \hat{x}(s+N-1)] \approx V_n^T \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}$$

Therefore, we can define the linear least squares problem:

$$\min_{\Phi_T, B_T, K_T} \| \begin{bmatrix} \hat{x}(s+1) & \hat{x}(s+2) & \cdots \end{bmatrix} - \begin{bmatrix} \Phi_T & B_T & K_T \end{bmatrix} \begin{bmatrix} \hat{x}(s) & \hat{x}(s+1) & \cdots \\ u(s) & u(s+1) & \cdots \\ y(s) & y(s+1) & \cdots \end{bmatrix} \|$$

Estimating Kalman from i/o data $\equiv$ solving 2 LSQ problems + 1 SVD

Next: Probing some future developments

Approximate H_2 optimal AO control