Homework exercise I

LA for Filtering and Identification

Preferably hand in your solutions as a single PDF file that also includes your m-files. Each student taking part in the exam has to submit one solution set.

Exercise 1: Partial Derivative

The entries of a matrix A depend on a parameter vector θ, expressed as $A(\theta)$. We have that $A \in \mathbb{R}^{n \times n}$ with $n > 1$. Show that the partial derivative of the inverse of A satisfies,

$$\frac{\partial A(\theta)^{-1}}{\partial \theta} = -A(\theta)^{-1}\frac{\partial A(\theta)}{\partial \theta}A(\theta)^{-1}$$

provided the inverses exist.

Exercise 2: Norm of pseudo-inverse

Let the pseudo inverse of a full rank matrix $A \in \mathbb{R}^{m \times n}$ (for $m \geq n$) be denoted by $A^{\dagger}$.

1. Check that the matrix $A^{\dagger}$ given as $(A^T A)^{-1} A^T$ indeed satisfies the conditions that define the pseudo-inverse and given as:

$$AA^{\dagger}A = A \quad A^{\dagger}AA^{\dagger} = A^{\dagger} \quad (AA^{\dagger})^T = AA^{\dagger} \quad (A^{\dagger}A)^T = A^{\dagger}A$$

2. Determine that $\|A^{\dagger}\|_2 = \frac{1}{\sigma_n}$ with σ_n the smallest singular value of the matrix A.

Exercise 3: Proof the LSQR-Theorem

This Theorem is given on slide 39 of Lecture 2 - day 1.

Exercise 4: Proof Exercise 2.12 p. 41

Exercise 5: Rank deficient LS problems

Consider the QR factorization with **pivoting** of a matrix $F \in \mathbb{R}^{m \times n}$ (for $m \geq n$), given as,

$$F\Pi = \underbrace{\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}}_{Q} \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \quad R_{11} \in \mathbb{R}^{r \times r} \text{ full rank and upper triangular} \quad r < n$$

The pivoting matrix Π is zero matrix except only one unit entry on each row. The matrix Q is orthogonal.

1. Show that the set $\mathcal{X}$ of all vectors x that minimize the norm $\|y - Fx\|_2$ is convex. [*Hint: It needs to be shown that for any two vectors x_1, x_2 belonging to $\mathcal{X}$ that $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{X}$ for $\lambda \in [0,1]$.*]
2. Use the result of 1 to show that the element of $\mathcal{X}$ that has minimal 2-norm is unique.
3. Parametrize the elements of the set $\mathcal{X}$ in terms of the given QR factorization with pivoting information.
4. Use the parametrization of 3 to determine the unique element of the set $\mathcal{X}$ with minimal 2-norm.

Exercise 6: Auto-correlation function calculation

Consider the following *stable* LTI system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

The input $u(k)$ is zero-mean white noise, with $E[u(k)u^T(k)] = \sigma^2 I$. Then $x(k)$ is a stationary stochastic signal, with $E[x(k)x^T(k)] = P$.
(a) Find an expression the steady-state value of the state covariance P.
(b) Compute:

$$R_n = E[y(k+n)y^T(k)].$$

for $n = 0, 1, \ldots$, as a function of A, B, C, D, and σ.
Hint: consider $n = 0$ and $n > 0$ separately.

References

[1] M. Verhaegen and V. Verdult, “Filtering and System Identification: A Least Squares Approach”, Cambridge University Press, 2007.